Previously

Monads on posets

$T : P \longrightarrow P$

$1 \leq T$

$T^2 \leq T$.

closure operators

Algebras for a monad

$Alg(T) \xrightarrow{i} P$

- i has a left adjoint $L : P \rightleftarrows Alg(T) : i$
- $T = i L$

Monads, the full story.

Def A monad on a category C is the data of (T, η, μ) where.

$T : C \longrightarrow C$ is an endofunctor

$\eta : 1 \Longrightarrow T$ is a natural trans

$\mu : T^2 \Longrightarrow T$ is a nat trans

such that the diagrams below commute.

$$
\begin{tikzcd}
T^3 \arrow[r, Rightarrow, "T(\mu)"] \arrow[d, Rightarrow, "\mu_T"'] & T^2 \arrow[d, Rightarrow, "\mu"] \\
T^2 \arrow[r, Rightarrow, "\mu"] & T
\end{tikzcd}
\qquad
\begin{tikzcd}
T \arrow[r, Rightarrow, "\eta_T"] \arrow[dr, "1"'] & T^2 \arrow[d, Rightarrow, "\mu"] \\
& T
\end{tikzcd}
\qquad
\begin{tikzcd}
T \arrow[r, Rightarrow, "T(\eta)"] \arrow[dr, "1"'] & T^2 \arrow[d, Rightarrow, "\mu"] \\
& T
\end{tikzcd}
$$

Rmk. Don't worry, just stay close to the posetal case. We will see why we care, and what the axioms mean.

Lemma $C \underset{U}{\overset{F}{\rightleftarrows}} D$ $\quad \varepsilon : FU \Rightarrow 1_C \quad \eta : 1_D \Rightarrow UF$

induces a monad on D given by

$$UF = T : D \longrightarrow D$$

Proof. We need to find

$$\left.\begin{array}{ll} (\eta) & 1 \Longrightarrow T \\ (\mu) & T^2 \Longrightarrow T \end{array}\right\} + \text{axioms to check.}$$

The unit is given in a trivial way. η

About the multiplication, we need $\mu : UFUF \Longrightarrow UF$

$$FU \xRightarrow{\varepsilon} 1_C$$

$\downarrow$ apply U on the left

$$UFU \xRightarrow{U(\varepsilon)} U$$

$\downarrow$ evaluate at F

$$\mu : UFUF \xRightarrow{U(\varepsilon)_F} UF$$

Now the axioms.

$$\begin{tikzcd} UF \arrow[r, Rightarrow] \arrow[dr, dash, "1"'] & UFUF \arrow[d, Rightarrow] & UF \arrow[l, Rightarrow] \arrow[dl, dash, "1"] \\ & UF & \end{tikzcd}$$

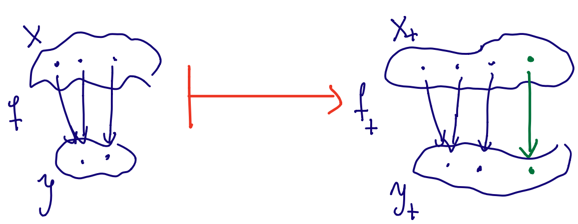
commutes by triangular identities of the adj.

$$\begin{tikzcd} UFUFUF \arrow[r, Rightarrow] \arrow[d, Rightarrow] & UFUF \arrow[d, Rightarrow] \\ UFUF \arrow[r, Rightarrow] & UF \end{tikzcd}$$

commutes by naturality of $U(\varepsilon)$

Rmk So, that's where the axioms come from. Of course, this does not tell what they mean...

Example Consider the endofunctor $(-)_+ : \mathrm{Set} \to \mathrm{Set}$. sending $X \longmapsto X \amalg \{\bullet\}$

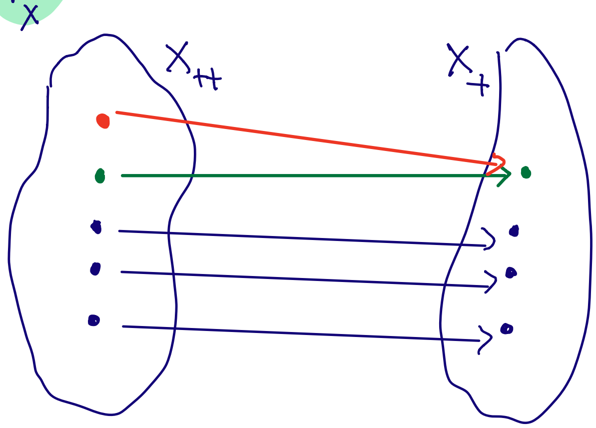


We shall prove that this is a monad.

$$\eta_X : X \longrightarrow X \amalg \{\bullet\}$$

is just the inclusion.

$$\mu_X : X \amalg \{\bullet\} \amalg \{\bullet\} \Longrightarrow X \amalg \{\bullet\}$$



What are the axioms here?

$$\begin{tikzcd}
A_{+++} \arrow[r, "(\mu_A)_+"] \arrow[d, "\mu_{(A_+)}"'] & A_{++} \arrow[d, "\mu_A"] \\
A_{++} \arrow[r, "\mu_A"'] & A_+
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "\eta_{A_+}"] \arrow[dr, dash, "1"'] & A_{++} \arrow[d, "\mu"] \\
& A_+
\end{tikzcd}$$

this is called the "maybe monad".

Example Consider the functor $R[-]: Set \longrightarrow Set$

$$X \longmapsto R[X]$$

$$R[X] = \left\{ \sum r_i x_i \quad r_i \in R \right\}.$$

R is a ring.

finite sums

You are very used to the case $R = \mathbb{K}$, a field, where this gives the free vector space.

This is a monad.

$$\eta_X : X \longrightarrow R[X]$$

is just the inclusion "in the base".

ring multiplication

$$\mu_X : R[R[X]] \longrightarrow R[X]$$

$$\sum_i \left[r_i \cdot \left(\sum_j r_j x_j \right) \right] \longmapsto \sum_i \sum_j (r_i \cdot r_j) x_j$$

Axioms of monad now are just saying that "evaluations go as expected".

So, why do we care about monads?

Well, we mainly care about their algebras

(we don't care about closure operators, they are tools to study closed sets!)

Def An algebra for a monad T on $\mathcal{C}$ is a couple (c, a) where

$$Tc \xrightarrow{a} c$$

c is an object and a is a morphism, such that:

$$\begin{tikzcd} Tc \arrow[r, shift left, "a"] & c \arrow[l, shift left, "\eta_c"] \end{tikzcd}$$

$$c \circ \eta_c = 1$$

$$\begin{tikzcd} T^2c \arrow[r, "\mu_c"] \arrow[d, "Ta"'] & Tc \arrow[d, "a"] \\ Tc \arrow[r, "a"] & c \end{tikzcd}$$

this is telling you that a is coherent with the multiplication.

Example pointed set = algebra for $( )_+$.

Example R-module = algebra for $R[\ ]$

Def A morphism of algebras is an f s.t.

$$\begin{tikzcd} T(c) \arrow[r, "Tf"] \arrow[d, "a_c"'] & T(d) \arrow[d, "a_d"] \\ c \arrow[r, "f"'] & d \end{tikzcd}$$

Def $\mathrm{Alg}(T)$ → algebras
→ morphisms of algebras.

Rem We saw that an adj induces a monad. Now we show that every monad can be broken into an adj.

$$\begin{tikzcd} \mathrm{Adj}(\mathcal{C}) \arrow[rr, bend right] & & \mathrm{Monad}(\mathcal{C}) \arrow[ll, bend right, dashed, "\mathrm{Alg}(-)"'] \end{tikzcd}$$

We already saw this phenomenon for posets.

Construction There is a functor (forgetful).

$$\begin{aligned} \mathcal{C} &\longleftarrow \mathrm{Alg}(T) : \mathcal{U}_T \\ c &\longmapsfrom c \\ f &\longmapsfrom f \end{aligned}$$

Rem the functor is faithful and conservative.

$$\boxed{\textbf{thm} \quad U_T \text{ has a left adjoint.}}$$

$$\begin{tikzcd} Alg(T) \arrow[d, "U_T"] \\ C \arrow[u, dashed, bend left, "F_T"] \end{tikzcd}$$

Proof. In the posetal case the strategy was to show that $T(x)$ is an algebra.
And this strategy still works.
Indeed the multiplication makes $T(x)$ into a natural algebra!

$$\boxed{\begin{matrix}\mu \\ \parallel \\ a\end{matrix}} : T(T(X)) \longrightarrow T(X)$$

So $F_T(X) = (Tx, \mu_X)$.

$F_T(f) = T(f)$.

Let us show that this is an adj.
(it is evident that $U_T F_T = T$)

We need to find

$$\eta : 1 \longrightarrow U_T F_T$$

$$\varepsilon : F_T U_T \longrightarrow 1_{Alg(T)}.$$

- η is given by the monad structure of T.

- ε, let's unpack it

$$\begin{tikzcd}
T^2(c) = F_T U_T F_T U_T(c) \arrow[r, "T_\varepsilon"] \arrow[d, "\mu_{Tc} = a_{F_T U_T c}"'] & T(c) \arrow[d, "a_c"] \\
T(c) = F_T U_T(c) \arrow[r, "\varepsilon"'] & c
\end{tikzcd}$$

We need an ε that makes this diagram commute. But this is just μ!

And then triangular qualities reduce to monad axioms.

<u>Discussion</u>

$$\begin{tikzcd}
 & \mathrm{Alg}(T) \arrow[dl] \arrow[d] \arrow[dd] \arrow[dr] \arrow[rr, bend right, "U_T"'] & & \mathcal{C} \arrow[ll, bend right, "F_T"'] \arrow[d] \\
\text{Monoids} & \text{Groups} & \text{Comp } T_2 \text{ spaces } \color{red}{(\beta, \text{monad})} & \text{Set.} \\
 & R\text{-modules} & &
\end{tikzcd}$$

Construction. Given an adj $\begin{tikzcd} D \arrow[r, bend right, "U"'] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend right, "F"'] \end{tikzcd}$

I can compute $Alg(UF)$.

$$\begin{tikzcd}
 & & Alg(UF) \arrow[d, "U_T"] \\
D \arrow[urr, dashed, "C"] \arrow[rr, "U"] & & C \arrow[ll, dashed, bend left, "F"] \arrow[u, dashed, bend right, "F_T"']
\end{tikzcd}$$

There exist a functor $C : D \longrightarrow Alg(UF)$ that "compares" D to the category of algebras for the induced monad.

Proof We show that $U(d)$ is an algebra for UF.

$$UFU(d) \xrightarrow{U(\varepsilon)_d} Ud$$

is the algebra structure

$$d \longmapsto (Ud, U(\varepsilon)_d).$$

Def $\mathcal{U} : \mathcal{D} \longrightarrow \mathcal{C}$ is monadic

iff

(1) has a left adj.

(2) the functor $\mathcal{C}$ in the construction above is an equivalence

thm (Beck) Let $\mathcal{U}: \mathcal{D} \longrightarrow \mathcal{C}$ be a functor. TFAE.

(1) $\mathcal{U}$ is monadic

(2)
- $\mathcal{U}$ has a left adjoint
- $\mathcal{U}$ is conservative
- $\mathcal{U}$ creates coequalizers of $\mathcal{U}$-split pairs!

CATEGORY THEORY

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EXERCISES

Exercise 1 (). Describe the monads (unit and counit) on Set whose algebras are: monoids, groups, semigroups.

Exercise 2 (). Consider the free-forgetful adjunction D : **Set** $\leftrightarrows$ Top : U, where D equips a set with the discrete topology over it. Compute the algebras for the induced monad over **Set**.

Exercise 3 (). A monad T on a category C is idempotent if its multiplication if its multiplication is an isomorphism. Show that the fortgetful functor U_T : Alg$(T) \to C$ of an idempotent monad is fully faithful.

Exercise 4 (). Let C be a category with coproducts and a terminal object. Can you always put a monad structure on the *maybe endofunctor* $c \mapsto c \coprod 1$?

Exercise 5 (). Show that the category of fields is not monadic over Set.

Exercise 6 (). Show that there is a monad on directed graphs whose algebras are small categories.

Exercise 7 (). Show that there is a monad on the category of small categories (and functors) whose algebras are posets.

The riddle of the week (,). Describe a monad structure on the functor of ultrafilter $X \mapsto \beta(X)$. Show that its algebras are compact T2 spaces (and continuous functions).

- the exercises in the red group are mandatory.
- pick at least one exercise from each of the yellow groups.
- pick at least two exercises from each of the blue groups.
- nothing is mandatory in the brown box.
- The riddle of the week. It's just there to let you think about it. It is not a mandatory exercise, nor it counts for your evaluation. Yet, it has a lot to teach.
- useful to deepen your understanding. Take your time to solve it. (May not be challenging at all.)
- measures the difficulty of the exercise. Note that a technically easy exercise is still very important for the foundations of your knowledge.
- It's just too hard.

The label **Leinster** refers to the book **Basic Category Theory**, by *Leinster*.
The label **Riehl** refers to the book **Category Theory in context**, by *Riehl*.

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