

FUNCTORIAL SEMANTICS 25

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rules

- Hand your exercises by the **end of the course** via email. In order to make my life easier, make sure to include the word **FUN25 in the subject**.
- Pick at least one exercise from each of the yellow groups.
- You must charge at least **2** full batteries!

Example. The vector of exercises [3,7,8,13,16] would pass this sheet.

EXERCISES

categories

Exercise 1 (□). How many idempotent monads $T : \mathbf{Set} \rightarrow \mathbf{Set}$ can you find on the category of sets? Describe them all.

Exercise 2 (□). Let $T : \mathbf{Set} \rightarrow \mathbf{Set}$ be a cocontinuous monad. Prove or provide a counterexample for the following statement: there exists a monoid M such that $T \cong M \times (-)$ as monads.

Exercise 3 (□). A *graph* (E, V, s, t) is the data of two sets E, V and two functions $s, t : E \rightrightarrows V$. Morphisms of graphs are defined as expected, and so is the category \mathbf{Gra} of graphs. Can you find a full subcategory C containing two objects such that every cocontinuous functor $\mathbf{Gra} \rightarrow \mathbf{Set}$ is uniquely determined by its value on C ?

Exercise 4 (□). Let \mathcal{A} be a cocomplete category and $a \in \mathcal{A}$ be a dense object, i.e. the family consisting of the single object a forms a dense generating set. Show that \mathcal{A} admits a faithful right adjoint $\mathcal{A} \rightarrow \mathbf{Set}$ and exhibit a category \mathcal{A} for which it is not an equivalence of categories.

Exercise 5 (□, ■). Let \mathcal{A} be a cocomplete category with a dense generating set. Show that \mathcal{A} is complete.

Exercise 6 (■). In the diagram below all the categories are λ -accessible and so are the functors f, g . Assume also that C is cocomplete. Justify that $\mathrm{lan}_g f$ exists and is accessible too.

$$\begin{array}{ccc}
 \mathcal{A} & \xrightarrow{f} & \mathcal{C} \\
 g \downarrow & \nearrow \mathrm{lan}_g f & \\
 \mathcal{B} & &
 \end{array}$$

universal algebra

Exercise 7 (□, ▢). Consider the categories \mathbf{Grp} , \mathbf{Ab} of groups and abelian group respectively.

- Describe the Lawvere theories axiomatizing them.
- Show that the inclusion $i : \mathbf{Ab} \hookrightarrow \mathbf{Grp}$ is a morphism of varieties.
- Describe the morphism of Lawvere theories that induce i .

Exercise 8 (□, ▢). Consider the categories \mathbf{Grp} , \mathbf{Ab} of groups and abelian group respectively.

- Describe the \mathbf{Set} monads axiomatizing them.
- Show that the inclusion $i : \mathbf{Ab} \hookrightarrow \mathbf{Grp}$ is a morphism of varieties.
- Describe the morphism of monads that induce i .

Exercise 9 (□, ▢). Recall that the category \mathbf{SLat} of suplattices is monadic over \mathbf{Set} . Following the standard construction that given a monad produces a (possibly large) algebraic theory, can you describe an equational presentation of the category of suplattices?

Exercise 10 (□). Let $T : \mathbf{Set} \rightarrow \mathbf{Set}$ be a finitary monad with some model with two distinct elements, show that its unit is injective.

Exercise 11 (▢). For every finitary monad $T : \mathbf{Set} \rightarrow \mathbf{Set}$ construct a finitary polynomial monad $P_T : \mathbf{Set} \rightarrow \mathbf{Set}$ and a morphism of monads $P_T \rightarrow T$ which is pointwise surjective.

sketches

Exercise 12 (□). Using the technology of the course show that every abelian group embeds in a divisible one.

Exercise 13 (□, ▢). Provide a sketch axiomatizing the category of fields. Could it be a limit sketch?

Exercise 14 (□, ▢). Given limit sketches S_1, S_2 define a symmetric tensor product $S_1 \otimes S_2$ in such a way that,

$$\mathbf{Mod}(S_1 \otimes S_2, \mathbf{Set}) \simeq \mathbf{Mod}(S_1, \mathbf{Mod}(S_2, \mathbf{Set})).$$

Exercise 15 (□). Show that the category of Banach spaces and non expansive maps is locally presentable. What about the category of Hilbert spaces?

Exercise 16 (□). Show that the category of topological spaces and the category of suplattices are not locally presentable.

Exercise 17 (▢). Show that if \mathcal{A} is locally finitely presentable, so are \mathcal{A}^\rightarrow and \mathcal{A}/a .

The riddle (Givant, ▲). A finitary monad $T : \mathbf{Set} \rightarrow \mathbf{Set}$ is *stable* if every algebra is free. (Assuming choice) show that there exactly four families of finitary stable monads $T : \mathbf{Set} \rightarrow \mathbf{Set}$. *Comment.* If you solve it with category theoretic methods, you can publish it.