Morley's Categoricity Theorem and Shelah's Categoricity Conjecture

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1 Introduction

In 1954, during a stagnant time the history of Model Theory, the mathematician Jerzy Łoś publishes a paper, [6], in which he gives the definition of κ-categoricity for a theory[1], and asks whether categoricity in any uncountable cardinal κ would imply categoricity in (i) any greater cardinal; (ii) any smaller (but still uncountable) cardinal. The question was deemed by him non-trivial and provided some examples in favour of it. Many mathematicians, including Łoś himself, approached the problem. Some notable early contributions are due to Vaught [17] and Ehrenfeucht [3, 4].

In 1962, the PhD student Michael Darwin Morley, working under the supervision of Vaught and Mac Lane, defended a thesis whose core was a positive answer to Łoś's question, and would later be known as *Morley's (Categoricity) theorem.* The thesis result was first published in the Proceedings of the AMS [8] and later in the Transactions of the AMS [7]. In his work, Morley proved a stronger version of the result, showing that the models arising from said theories are saturated, meaning that the model satisfies all complete types of "reasonable" size. At the time it was already well-known that two saturated models of the same cardinality (the word used at the time was "power") are isomorphic, leading to the result.[2]

Following Morley's great work a new question naturally arises, namely "How can we generalize this result to uncountable theories?". According to Saharon Shelah, in a paper published in 1974, [12], some first, independent, approaches to this question are due to Rowbottom [10], Ressayre [9], and Shelah [11, 15]. In the same paper, he generalizes Łoś' conjecture to

Conjecture. *Let T be a first order theory. Let $\kappa > |T|$ be a cardinal. If T is κ-categorical, then it is μ-categorical for all $\mu > |T|$.*

Łoś' conjecture is obtained when T is a countable theory. In [12], the author provides a proof for the conjecture, which will later be known as *Shelah's (Categoricity)*

[1] This was given by Vaught indipendently and at the same time in [16].

[2] For more on this see [1]

Conjecture. In it, the notion of stability (on which *Stability Theory* is built) is introduced and the paper lays the groundwork for the subsequent *Classification theory*, a program in which Shelah aimed to categorize theories for which models can be classified in some combinatorial fashion and theories too rich of models for a classification to even be possible. Many mathematicians, Shelah included, worked on this project and many subsequent approximations were produced, as well as a lot of definitions introduced. Most famously the notions of super-stability (for a theory), abstract elementary classes (AEC), and forking, an extension of the concept of independence to types, which was introduced (in a combinatorial fashion) by Shelah in his first edition of his book *Classification Theory and the Number of Non-isomorphic Models* [13]. Eventually, the goal was reached in 1990[3], in the second edition (published 12 years after the first) of the same book [14].

2 Technologies

2.1 Morley's theorem

In [7], Morley introduced the concept of **space of types** (although he never uses this name, and refers to it as $S(A)$), heavily used the notion of **saturated models**, for which the result of essential uniqueness was already known, employed the **back-and-forth** method and the **omitting type theorem**. An entire section is dedicated to results depending on **Ramsey's theorem**. Finally, he uses the work of Ehrenfeucht and Mostowski [5] on **sets of indiscernibles**, i.e. elements that cannot be told apart only using first order formulas, to construct non-isomorphic models in totally transcendental theories.

The main contributors to Morley's theorem are:

1. Jerzy Łoś: Formulated the initial problem, as well as some initial examples suggesting why the conjecture could hold.
2. Andrzej Ehrenfeucht and Andrzej Mostowski: established core results on indiscernibles later employed by Morley in his work.
3. Robert Lawson Vaught: made some first approximations of the affirmative result. Supervised Morley's work.
4. Michael Darwin Morley: Affirmatively answered Łoś' question.

2.2 Shelah's conjecture

In [12], Shelah generalized the Łoś conjecture to **uncountable theories** and established a positive result to the SCC. In his work he accurately generalized the procedure laid down by Morley in [7], introducing **stability** as a generalization of **ω-stability**, localizing many notions, most importantly that of **saturation** of a model. He introduced a

[3]For more on this see [2]

notion of **degree on formulas** that he proved to be finite whenever a theory is categorical in some cardinal greater than the power of the theory. From this work, **Stability theory** and **Classification theory** are born.

The main contributors to Shelah's conjecture are:

-1. Jerzy Łoś: formulated the initial conjecture later generalized by Shelah.

0. Michael Darwin Morley: devised the structure to attack the problem in the smaller setting. Shelah later generalized his proof to the generalized environment.

1. Frederick Rowbottom and Jean-Pierre Ressayre: worked, individually, on the same problem.

2. Saharon Shelah: formulated and answered the generalized conjeture.

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