Seminar on Uemura's General Framework

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April 11, 2025

Abstract

In this talk, we will give an overview of Uemura's paper *A general framework for the semantics of type theory* [Uem23], as part of the course *Denotational Semantics of Dependent Type Theory* by Ivan Di Liberti.

Uemura's paper [Uem23] serves as general reference for everything we write below, even if we adapt some proofs.

1 Discrete fibrations

For $n \geq 0$ we will write $[n]$ for the poset category $\{0 < \ldots < n\}$. Recall:

Definition 1.1. A ***discrete fibration*** is a functor with unique (up to equality) right lifting against the right endpoint inclusion $1\colon [0] \to [1]$. Given a category $\mathcal{S}$, we write $\mathbf{Disc}_{/\mathcal{S}}$ for the full subcategory of $\mathcal{C}\mathrm{at}_{/\mathcal{S}}$ whose objects are discrete fibrations (with target $\mathcal{S}$). △

Lemma 1.2. (Grothendieck construction) *There is an equivalence*

$$\mathbf{Disc}_{/\mathcal{S}} \simeq \mathrm{Fun}(\mathcal{S}^{\mathrm{op}}, \mathsf{Set}), F \mapsto (s \mapsto F^{-1}(s))$$

of categories, whose inverse is given by sending a presheaf F to the projection map $\int F \to \mathcal{S}$ from its category of elements.

Remark 1.3. In particular, the representable presheaf $\text{よ}s$ corresponds to the discrete fibration $\mathcal{S}_{/s} \to \mathcal{S}$. We call these discrete fibrations ***representable***. ♢

Lemma 1.4. *A discrete fibration is representable iff it has a terminal object.* □

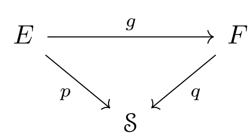
Corollary 1.5. (Yoneda lemma for discrete fibrations) *For every $s \in \mathcal{S}$ and every discrete fibration p over $\mathcal{S}$ there is an isomorphism*

$$\mathbf{Disc}_{/\mathcal{S}}(\mathcal{S}_{/s}, p) \cong p(s), f \mapsto f(\mathrm{id}_s).$$

We can (and will) therefore denote maps $\mathcal{S}_{/s} \to p$ with the corresponding element of $p(s)$. □

We saw that a natural model on a category $\mathcal{S}$ (possibly with the assumption that it has particular finite limits) is a representable map of presheaves over $\mathcal{S}$. Under the Grothendieck construction, this becomes a morphism of discrete fibrations.

Lemma 1.6. *Suppose given a map*



of discrete fibrations over $\mathcal{S}$. Then g has a right adjoint iff the corresponding map of presheaves is representable.

Proof. The map of presheaves corresponding to g is representable iff for every $f\colon \mathcal{S}_{/s} \to F$ there exists an object $h(f) \in E$ and a pullback diagram

$$\begin{array}{ccc} \mathcal{S}_{/ph(f)} & \xrightarrow{h(f)} & E \\ \downarrow & \lrcorner & \downarrow q \\ \mathcal{S}_{/q(f)} & \xrightarrow[f]{} & F \end{array} \tag{1}$$

Since p and q are discrete fibrations, the canonical maps $E_{/h(f)} \to \mathcal{S}_{/ph(f)}$ and $F_{/f} \to \mathcal{S}_{/q(f)}$ are equivalences of categories. Therefore the above pullback diagram takes the form

$$\begin{array}{ccc} E_{/h(f)} & \longrightarrow & E \\ \downarrow & \lrcorner & \downarrow q \\ F_{/f} & \longrightarrow & F \end{array}$$

and this says that all objects $h(f)$ assemble into a right adjoint functor to g. □

Definition 1.7. A morphism of discrete fibrations over $\mathcal{S}$ is ***representable*** if it has a right adjoint. △

Remark 1.8. Given a discrete fibration p on a category $\mathcal{S}$ *with a terminal object*, the unique morphism $p \to \mathrm{id}_{\mathcal{S}}$ of discrete fibrations is representable iff the discrete fibration p itself is representable in the sense of Remark 1.3. Since we will exclusively deal with the case where $\mathcal{S}$ has a terminal object, this terminology therefore should not cause confusion: the absolute notion of representability can be recovered from the relative one when mapping into the terminal object, as expected. ◊

Consider the situation of Lemma 1.6, write δ^g for the right adjoint of g, and suppose given an object $b \in F$ living over $s \in \mathcal{S}$. Then we denote by $\{b\}^g \in \mathcal{S}$ the object $p\delta^g(b)$. Diagram (1) gives us a map $\{b\}^g \to s$ in $\mathcal{S}$, that we think of as the context extension of s by b.

2 Representable map categories

Constructing a natural model on a category $\mathcal{S}$ is therefore picking out a representable map of discrete fibrations over $\mathcal{S}$. We will turn the act of picking out this morphism into a morphism of appropriate structures.

Recall:

Definition 2.1. An arrow $f\colon a \to b$ in a category $\mathcal{R}$ with finite limits is ***exponentiable*** if the pullback functor $f^*\colon \mathcal{R}_{/b} \to \mathcal{R}_{/a}$ has a right adjoint f_*. △

Definition 2.2. (Representable map category)

(i) A ***representable map category*** is a category $\mathcal{R}$ with finite limits, equipped with the data of a ***stable class of exponentiable maps***, which is a class $\mathcal{A}$ of arrows in $\mathcal{R}$ satisfying:

  (a) every identity morphism is in $\mathcal{A}$;
  (b) $\mathcal{A}$ is stable under composition and arbitrary pullbacks in $\mathcal{R}$;
  (c) every element of $\mathcal{A}$ is exponentiable.

  We will call an element of $\mathcal{A}$ a ***representable map***.

(ii) A ***representable map functor*** $F\colon \mathcal{R} \to \mathcal{S}$ is a functor between two representable map categories preserving finite limits, representable maps and pushforwards: for any representable $f\colon a \to b$ in $\mathcal{R}$, the canonical natural transformation

$$\begin{array}{ccc} \mathcal{R}_{/a} & \xrightarrow{f_*} & \mathcal{R}_{/b} \\ {\scriptstyle F}\downarrow & \swarrow & \downarrow{\scriptstyle F} \\ \mathcal{S}_{/Fa} & \xrightarrow[(Ff)_*]{} & \mathcal{S}_{/Fb} \end{array}$$

is invertible.

We write $\mathfrak{Rep}$ for the sub-2-category of $\mathfrak{Cat}$ on representable map categories and representable map functors. △

Example 2.3. (i) Given a category $\mathcal{S}$, the representable maps from Definition 1.7 turn $\mathbf{Disc}_{/\mathcal{S}}$ into a representable map category. (There are proof obligations here.)[1]

(ii) If $\mathcal{R}$ is a representable map category, then every slice $\mathcal{R}_{/a}$ is again a representable map category, in which an arrow is representable when the corresponding arrow in $\mathcal{R}$ is. We obtain a functor

$$\mathcal{R}_{/(-)} \colon \mathcal{R}^{\mathrm{op}} \to \mathfrak{Rep}$$

(with functoriality given by pullback).

(iii) If $\mathcal{R}$ is a category with finite limits and E is some class of exponentiable arrows in $\mathcal{R}$, we can form the smallest pullback stable class $\mathcal{A}$ containing E, which turns $\mathcal{R}$ into a representable map category.

▽

Definition 2.4. (Type theory) A ***type theory*** is a small representable map category. △

This definition lives very much in the world of categorical semantics, and less in the world of syntactic considerations. It is also not meant to capture all type theories people have come up with over the years, just a particular subclass that has enjoyed some research interest.

Example 2.5. (Basic dependent type theory) Let $\mathbb{G}$ be the opposite of the category of finite GATs and interpretations between those. We let U_0 and E_0 be the theories

$$\{\vdash A\}, \text{ and } \{\vdash A, \vdash a : A\}$$

respectively. There is an evident arrow $E_0 \to U_0$. Under Example 2.3(iii) this turns $\mathbb{G}$ into a representable map category, and a type theory in the sense of Definition 2.4 we call ***basic dependent type theory***. ▽

The above representable map category $\mathbb{G}$ satisfies an interesting universal property, tying it to the notion of a natural model.

Theorem 2.6. ([Uem23, 4.13]) *Given a representable map f in a representable map category $\mathcal{R}$, there is an essentially unique representable map functor $\mathbb{G} \to \mathcal{R}$ sending $E_0 \to U_0$ to f.*

3 Models of type theories

Convention 3.1. We will from now on consider $\mathbb{T}$ to be a type theory. ⊙

We find that a natural model on a category $\mathcal{S}$ is just the data of a representable map functor $\mathbb{G} \to \mathbf{Disc}_{/\mathcal{S}}$. We can define a model for $\mathbb{T}$ in an analogous way.

[1] That a representable map of fibrations is exponentiable follows from the fact that the slice functor $\mathbf{Disc}_{/(-)} \colon \mathfrak{Cat}^{\mathrm{op}} \to \mathfrak{Cat}$ with functoriality by pullback is a 2-functor and hence preserves adjunctions. That these representable maps are stable under arbitrary pullback follows from pullback pasting.

Definition 3.2. (Model) A ***model*** of $\mathbb{T}$ is a category $\mathcal{S}$ with terminal object, and a representable map functor $\mathbb{T} \to \mathbf{Disc}_{/\mathcal{S}}, A \to A^{\mathcal{S}}$. △

Note that the terminal object $1 \in \mathbb{T}$ is sent towards the identity functor on $\mathcal{S}$, and in this sense we can recover $\mathcal{S}$ from second part of the data of a model of $\mathbb{T}$.

We obtain a 2-category of models in the following way.

Definition 3.3. Given an isomorphism-commutative square of categories

$$\begin{array}{ccc} A' & \xrightarrow{u} & A \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ B' & \xrightarrow[v]{} & B \end{array}$$

such that f and f' have right adjoints g and g', we say that the ***Beck-Chevalley condition*** is satisfied if the canonical natural transformation $ug' \to gu$ is an isomorphism. △

Definition 3.4. Suppose given two models $(-)^{\mathcal{R}}$ and $(-)^{\mathcal{S}}$ of $\mathbb{T}$.

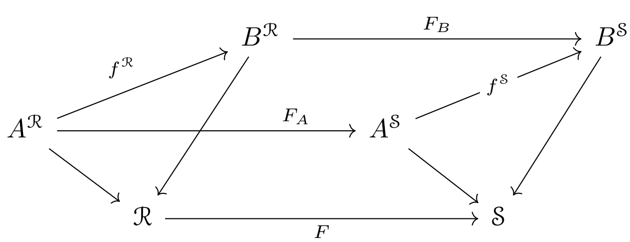
(i) A ***(1-)morphism*** from $(-)^{\mathcal{R}}$ to $(-)^{\mathcal{S}}$ is

(a) a functor $F\colon \mathcal{R} \to \mathcal{S}$ preserving terminal objects;

(b) for each $A \in \mathbb{T}$ an isomorphism-commutative square

$$\begin{array}{ccc} A^{\mathcal{R}} & \xrightarrow{F_A} & A^{\mathcal{S}} \\ \downarrow & & \downarrow \\ \mathcal{R} & \xrightarrow[F]{} & \mathcal{S} \end{array}$$

natural in A: for each $f\colon A \to B$ the diagram

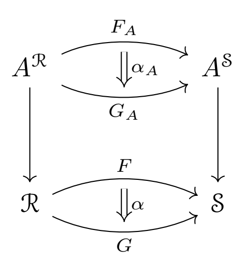


commutes up to isomorphism. Moreover, if f is representable, we require the top square to satisfy the Beck-Chevalley condition.

(ii) Suppose given two morphisms $F_{(-)}$ and $G_{(-)}$ from $(-)^{\mathcal{R}}$ to $(-)^{\mathcal{S}}$. A ***2-morphism*** $\alpha_{(-)}\colon F_{(-)} \to G_{(-)}$ consists of

(a) a natural transformation $\alpha\colon F \to G$ of functors $\mathcal{R} \to \mathcal{S}$;

(b) for each $A \in \mathbb{T}$ a natural transformation $\alpha_A\colon F_A \to G_A$ such that the diagram



commutes up to isomorphism. (Note that such an α_A is necessarily unique.)

We write $\mathfrak{Mod}_{\mathbb{T}}$ for the 2-category of models of $\mathbb{T}$. △

Note that we can recover F from F_A by taking A the terminal object of $\mathbb{T}$.

4 Democratic models

Recall the notion of context extension morphism $\{b\}^g \to s$ in $\mathcal{S}$ for $g\colon E \to F$ a representable map of discrete fibrations, and $b \in F$ living over s. The idea of a democratic model is that all objects of category in which the model lives can be thought of as types and context extensions.

Definition 4.1. (Democratic model) Let $A \mapsto A^{\mathcal{S}}$ be a model of $\mathbb{T}$. The class of ***contextual objects*** in $\mathcal{S}$ is generated inductively as follows:

(i) the terminal object is contextual;

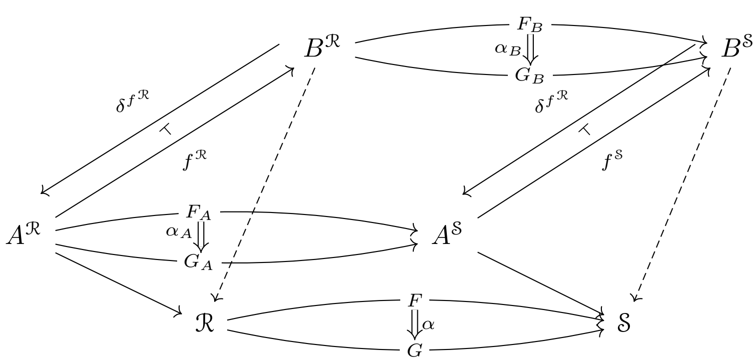
(ii) every object isomorphic to a contextual object is contextual;

(iii) if $b \in B^{\mathcal{S}}$ and a representable map $f\colon A \to B$ are given, then if B is contextual so is the context extension $\{b\}^f$.

The model $(-)^{\mathcal{S}}$ is ***democratic*** if all objects of $\mathcal{S}$ are contextual. We write $\mathfrak{Mod}_{\mathbb{T}}^{\mathrm{dem}}$ for the full sub-2-category of $\mathfrak{Mod}_{\mathbb{T}}$ on democratic models. △

Lemma 4.2. *If $\mathcal{R}$ and $\mathcal{S}$ are models of $\mathbb{T}$ and $\mathcal{R}$ is democratic, then $\mathfrak{Mod}_{\mathbb{T}}(\mathcal{R}, \mathcal{S})$ is a discrete category. In particular, the 2-category $\mathfrak{Mod}_{\mathbb{T}}^{\mathrm{dem}}$ is a 1-category.*

Proof. We sketch the proof: if $\alpha\colon F \to G$ is a 2-morphism of models, we wish to show $\alpha(I)\colon FI \to GI$ is uniquely determined and invertible for each contextual object I of $\mathcal{R}$. We argue inductively. When I is terminal this follows from FI and GI both being terminal. For the induction step, suppose the statement holds for I. Pick a representable $f\colon A \to B$ in $\mathbb{T}$ and $b \in B^{\mathcal{R}}(I)$. We want to show the statement also holds for $\{b\}^f$. The diagram



commutes up to isomorphism (using the Beck-Chevalley condition for the top squares). Starting with $b \in B^{\mathcal{R}}(I)$, this gives us a commutative diagram

$$\begin{array}{ccc} F\{b\}^f & \xrightarrow{\cong} & \{F_B b\}^f \\ {\scriptstyle \alpha(\{b\}^f)}\downarrow & & \downarrow{\scriptstyle \{\alpha_B(b)\}^f} \\ G\{b\}^f & \xrightarrow{\cong} & \{G_B b\}^f \end{array}$$

Moreover, $\alpha_B(b)$ is the cartesian lift of $\alpha(I)$ ending at $G_B b$, so is uniquely determined by our induction hypothesis. If $\alpha(I)$ is invertible, so is $\alpha(\{b\}^f)$ by running the same argument in reverse order on $\alpha(I)^{-1}$. Finally, the maps α_C for $C \in \mathbb{T}$ are uniquely determined by α, as we noted earlier, so this fully determines the 2-morphism of models. □

Every model has a largest democratic submodel:

Construction 4.3. If $\mathcal{S}$ is a model of $\mathbb{T}$, then the ***heart*** of this model is the model $\mathcal{S}^{\heartsuit}$ whose underlying category is the full subcategory of $\mathcal{S}$ on contextual objects, and which sends $A \in \mathbb{T}$ to the pullback $A^{\mathcal{S}^{\heartsuit}} := \mathcal{S}^{\heartsuit} \times_{\mathcal{S}} A^{\mathcal{S}}$. △

The following lemma follows inductively from the definition of contextual objects.

Lemma 4.4. *For any models $\mathcal{R}$ and $\mathcal{S}$ of $\mathbb{T}$ with $\mathcal{R}$ democratic, the inclusion $\mathcal{S}^{\heartsuit} \hookrightarrow \mathcal{S}$ induces a natural equivalence*

$$\mathfrak{Mod}_{\mathbb{T}}(\mathcal{R}, \mathcal{S}^{\heartsuit}) \to \mathfrak{Mod}_{\mathbb{T}}(\mathcal{R}, \mathcal{S}).$$

5 The initial model of a type theory

Using this, we can construct the initial model of a type theory.

Definition 5.1. (The (bi-)initial model) The Yoneda embedding $\mathbb{T} \to \mathbf{Disc}_{/\mathbb{T}}$ defines a model of $\mathbb{T}$. We denote its heart by $\mathcal{I}(\mathbb{T})$, and call it the ***bi-initial model***. △

Theorem 5.2. *The model $\mathcal{I}(\mathbb{T})$ is the initial object (in 2-categorical sense) of the 2-category $\mathfrak{Mod}_{\mathbb{T}}$.*

Proof. We will sketch the proof. First we build a functor $\mathcal{I}(\mathbb{T}) \to \mathcal{S}$ for any model $\mathcal{S}$ of $\mathbb{T}$, as follows: for every $A \in \mathcal{I}(\mathcal{S})$. the map $A^{\mathcal{S}} \to \mathcal{S}$ of discrete fibrations over $\mathcal{S}$ is representable since A is contextual, so since $\mathcal{S}$ has a terminal object $A^{\mathcal{S}}$ is a representable discrete fibration. We let $F : \mathcal{I}(\mathbb{T}) \to \mathcal{S}$ be the functor sending A to the representing object, and for $B \in \mathbb{T}$ we let $F_B : B^{\mathcal{I}(\mathbb{T})} \to B^{\mathcal{S}}$ be the map $\mathcal{I}(\mathbb{T})_{/B} \to B^{\mathcal{S}}$ sending $f : A \to B$ (with $A \in \mathcal{I}(\mathcal{S})$) towards the element of $B^{\mathcal{S}}(FA)$ classified by

$$\mathcal{S}_{/FA} \simeq A^{\mathcal{S}} \xrightarrow{f^{\mathcal{S}}} B^{\mathcal{S}}.$$

This turns out to be a morphism of models. For unicity of this map, suppose G is another such map. Then the square

$$\begin{array}{ccc} \mathcal{I}(\mathbb{T})_{/A} & \xrightarrow{G_A} & A^{\mathcal{S}} \\ \downarrow & & \downarrow \\ \mathcal{I}(\mathbb{T}) & \xrightarrow{G} & \mathcal{S} \end{array}$$

commutes and satisfies the Beck-Chevalley condition, and the map $\mathcal{I}(\mathbb{T}) \to A^{\mathcal{S}}$ preserves a terminal object. Therefore $A^{\mathcal{S}}$ is representable by GA. For $B \in \mathbb{T}$ and $f : A \to B$ we have a commutative diagram

$$\begin{array}{ccc} \mathcal{I}(\mathbb{T})_{/A} & \xrightarrow{G} & \mathcal{S}_{/GA} \\ & \searrow^{G_A} & \downarrow \cong \\ \downarrow f & & A^{\mathcal{S}} \\ & & \downarrow f^{\mathcal{S}} \\ \mathcal{I}(\mathbb{T})_{/B} & \xrightarrow[G_B]{} & B^{\mathcal{S}} \end{array}$$

which shows $G_B(f)$ is forced to be defined like $F_B(f)$ was. □

6 Theories and internal languages

Definition 6.1. (Theory) A ***theory*** over a type theory $\mathbb{T}$ (also called a $\mathbb{T}$-***theory***) is a functor $K\colon \mathbb{T} \to \mathsf{Set}$ that preserves finite limits. We let $\mathbf{Th}_{\mathbb{T}}$ be the 1-category of $\mathbb{T}$-theories and natural transformations between them. △

Intuitively, for $A \in \mathbb{T}$ the set $K(A)$ is the set of closed derivations of judgment form A. Note that we do not require K to preserve representable maps, because if it did then the set of maps from $K(A)$ to $K(B)$ would need to be isomorphic to $K(A \Rightarrow B)$ if our type theory is basic dependent type theory. This however does not align with the intuitive understanding of the sets $K(-)$ that we want to have.

Example 6.2. If $\mathbb{T} = \mathbb{G}$, basic dependent type theory, then $\mathbb{T}$-theories are precisely GATs. For a theory K, the set $K(U_0)$ is the set of closed types in our GAT and $K(E_0)$ is the set of closed terms. ▽

Definition 6.3. (Internal language) Given a model $(\mathcal{S}, (-)^{\mathcal{S}})$ of $\mathbb{T}$, the ***internal language*** of $\mathcal{S}$ is the $\mathbb{T}$-theory $\mathbb{L}_{\mathbb{T}}\mathcal{S}$ defined by $\mathbb{L}_{\mathbb{T}}\mathcal{S}(A) = A^{\mathcal{S}}(1)$, where 1 is the terminal object of $\mathcal{S}$. △

Equivalently, $\mathbb{L}_{\mathbb{T}}\mathcal{S} \cong \mathbf{Disc}_{/\mathcal{S}}(\mathcal{S}, (-)^{\mathcal{S}})$, where $\mathcal{S}$ is the identity discrete fibration over $\mathcal{S}$. This makes it clear that we have a functor $\mathbb{L}_{\mathbb{T}}\colon \mathfrak{Mod}_{\mathbb{T}} \to \mathbf{Th}_{\mathbb{T}}$. We will not prove the following, but given a $\mathbb{T}$-theory K, we can build a new type theory $\mathbb{T}[K]$, intuitively adding to our type theory new types and terms corresponding to what we find in K.[2] We can then form the initial model $\mathcal{I}(\mathbb{T}[K])$ of it and let $\mathbf{M}_{\mathbb{T}}(K)$ be the model given by

$$\mathbb{T} \to \mathbb{T}[K] \xrightarrow{(-)^{\mathcal{I}(\mathbb{T}[K])}} \mathbf{Disc}_{/\mathcal{I}(\mathbb{T}[K])}.$$

Theorem 6.4. *The assignment $K \mapsto \mathbf{M}_{\mathbb{T}}(K)$ extends to a 2-functor fitting in a 2-adjunction*

$$\mathbf{M}_{\mathbb{T}}\colon \mathbf{Th}_{\mathbb{T}} \rightleftarrows \mathfrak{Mod}_{\mathbb{T}} \colon \mathbb{L}_{\mathbb{T}}.$$

This 2-adjunction restricts to inverse equivalences

$$\mathbf{M}_{\mathbb{T}}\colon \mathbf{Th}_{\mathbb{T}} \rightleftarrows \mathfrak{Mod}_{\mathbb{T}}^{\mathrm{dem}} \colon \mathbb{L}_{\mathbb{T}}$$

of 1-categories.

References

[Uem23] T. Uemura. "A general framework for the semantics of type theory". In: *Math. Struct. in Comp. Science* 33 (2023), pp. 134–179.

[2]Formally, $\mathbb{T}[K]$ is the pseudocolimit of the composite $(\int_{\mathbb{T}} K)^{\mathrm{op}} \to \mathbb{T} \xrightarrow{\mathbb{T}_{/(-)}} \mathfrak{Rep}$.