L14 Topoi as objects.

Hello and welcome to the third lecture of this module on topos theory. The menu for this lecture is

- Give (finally) the defn of topos)
- Discuss concrete ways to present a topos
  - G. topologies
  - LT topologies
- Introduce the 2-category of topoi.

**Def** (topos) A topos is a lex-reflective subcategory of a presheaf category.

$$\begin{tikzcd} \mathcal{E} \arrow[r, hook, "j"', shift right] & \mathrm{Psh}(\mathcal{C}) \arrow[l, "l"', shift right] \arrow[l, phantom, "\perp"] \end{tikzcd}.$$

**Rem** Localic topoi are topoi.

**Rem** Fin is not a topos.

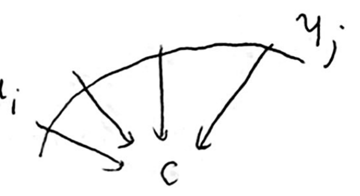
**Part 1** Presenting topoi. The definition above is perfectly fine, and a lot (if not all) of topos theory can be developed with that definition. Yet, in many concrete circumstances one wants to present a topos via some concrete data. To some extent one can say that a locale is a compact presentation of a localic topos.

In this spirit, we shall now see some ways to specify topoi:

$\boxed{1}$ Grothendieck topologies.

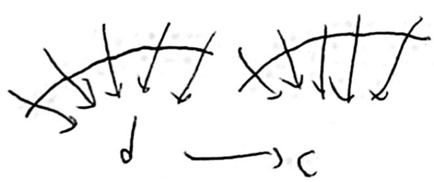
The first way to present a topos is strongly inspired by the localic one.

<u>Def</u> Let $\mathcal{C}$ be a category with finite limits. A G. topology is, for every object $c$ in $\mathcal{C}$, the data of



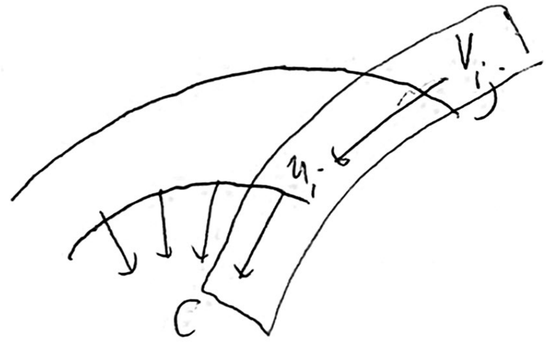

families of maps into $c$ (covering families) with the properties that:

(1) we can pull back families




(2) Isos cover

(3) covering families are nestable




<u>Rem</u> Of course a locale has a canonical topology given by covering families.

<u>Rem</u> Notice that we could already present a notion of sheaf w.r.t. such topology by mimicking the localic definition.

**Rem** The assumption that $\mathcal{C}$ has finite limits is not necessary, but somewhat useful, especially if we want to copy paste the def of sheaf

$$P(U) \longrightarrow \prod P(U_i) \rightrightarrows \prod P(\underbrace{U_i \cap U_j}_{\text{this is a pullback.}})$$

$\boxed{2}$ A more categorical pov.

Now notice that the collection of covering families over a specific object gives a functorial assignment

$$J : \mathcal{C}^{op} \longrightarrow Set$$

$$c \longmapsto \{ \text{covering families over } c \}$$

where the functoriality is given by axiom (1) in the definition of topology. Moreover, every covering family is by definition a set

$$\boxed{S \subset よ(c)}$$ a subobject of a representable

so that

$$J(c) \rightarrowtail Sub(よ(c)).$$

Now, if you remember our previous lecture, $Sub(よ(-))$ is nothing but the subobject classifier in $Psh(\mathcal{C})$.

So, another approach to topologies is to specify

A subobject $J \hookrightarrow \Omega$ in $Psh(\mathcal{C})$ with the additional properties that

(•)

$$\begin{tikzcd} & 1 \arrow[dl, dashed] \arrow[d] \\ J \arrow[r, hook] & \Omega \end{tikzcd}$$

1 factors through $J$

$\boxed{\text{Axiom (2)}}$

(•) Given a diagram when $m$ is in $J(c)$

$$\begin{tikzcd} S \arrow[r] \arrow[d, tail, "m"] & J \arrow[d] \\ y(c) \arrow[r, "F"] & \Omega \end{tikzcd}$$

$\exists!$ filler

$$\begin{tikzcd} S \arrow[r] \arrow[d] & J \arrow[d] \\ y(c) \arrow[r] \arrow[ur, dashed, "\exists!"] & \Omega \end{tikzcd}$$

$\boxed{\text{Axiom 3}}$

Now notice that to be a sheaf means exactly that for every covering family $S \rightarrowtail y(c)$ we have that

$$\begin{tikzcd} S \arrow[r] \arrow[d, "m"'] & P \\ y(c) \arrow[ur, dashed, "\exists!"] & \end{tikzcd}$$

So a sheaf is, by definition a presheaf that is orthogonal to all the pullbacks

$$\begin{tikzcd} S \arrow[r] \arrow[d, "m_F"'] \arrow[dr, phantom, "\lrcorner", very near start] & J \arrow[d] \\ y(c) \arrow[r, "F"] & \Omega \end{tikzcd}$$

$$Sh(\mathcal{C}, J) = \{m_F\}^{\perp}.$$

this orthogonality class is stable under p.b and so

We have a lex reflection.

## Recoup

— Grothendieck topology

Covering families | $P(U) \longrightarrow \prod P(U_i) \longrightarrow \prod P(U_i \cap U_j)$ notion of sheaf

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$J \hookrightarrow \Omega$ "covering families" | $P \perp$ to all the pullbacks notion of sheaf $*$

From $(*)$ we derive that $Sh(\mathcal{C}, J)$ is lex reflective and thus a topos.

**Def** A couple $(\mathcal{C}, J)$ is called site!

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## LT topologies.

Lawvere - Tierney topologies are yet another way to encapsulate the notion of topology.

Back to our previous notion $J \hookrightarrow \Omega$, we observe that - of course - such a mono is classified by a map

$$j : \Omega \longrightarrow \Omega.$$

with the property that

- $jx \geq x$
- $j(x \wedge y) = j(x) \wedge j(y)$
- $j^2 x = j$
- $j(1) = 1.$

this is called a Lawvere Tierney topology.

$$\boxed{\text{Groth topologies } J} \overset{\text{classifying map}}{\rightsquigarrow} \boxed{\text{LT topologies}}$$

One can also go in the opposite direction by

$$\begin{tikzcd}
J \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d] \\
\Omega \arrow[r, "j"] & \Omega
\end{tikzcd}$$

pulling back the operator.

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Some observation on LT topologies.

(1) We can find the subobject classifier $\Omega_j$ of the topos associated to $Sh(\mathcal{C}, J)$ by taking the epi-mono factorization of $j$.

$$\begin{tikzcd}
\Omega \arrow[rr, "j"] \arrow[dr, two heads] & & \Omega \\
& \Omega_j \arrow[ur, hook] &
\end{tikzcd}$$

(2) There is a very compact way to sheafify a presheaf, when we are given $\Omega_j$. (Due to Lawvere).

$$\begin{tikzcd}
P \arrow[r, two heads] \arrow[d, "c"'] & I_P \arrow[d, hook] \\
\Omega^P \arrow[r] & \Omega_j^P
\end{tikzcd}$$

OK, so we now have many ways to specify a topos. Of course they are even testable.
if $\mathcal{E}$ is a topos, I can define a LT topology in it and that will give me a lex reflector that finds a new topos

$$\begin{tikzcd} \mathcal{E}_j \arrow[r, hook, shift right] \arrow[r, phantom, "\perp"] & \mathcal{E} \arrow[l, bend right] \end{tikzcd}$$

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A 2-category of topoi

Discussion of geometric morphism induced by a continuous function

The notion of geometric morphism and the 2-cat of topoi

Morphism of sites and associated geometric morphism.