

L15 Classifying topoi.

The plan for the day is the following.

- (1) Introduce the notion of geometric theory.
- (2) Introduce syntactic categories.
- (3) Classifying topoi for geometric theories.

(1) Geometric theory.

We have discussed in the lectures on toposes that one can use "Boolean algebras" (aka fibred posets) to represent theories.

$\perp, \top, \neg, \vee, \wedge, \Rightarrow$ \mathcal{B} Boolean algebra.

Of course the algebraic structure \mathcal{B} comes equipped with reflects a kind of logic/deduction system we can do about.

Geometric logic is the logic of frames

$\perp, \top, \vee, \wedge$ \mathcal{L} frame

In this spirit, a model of a frame is precisely a point, and this is the great realization that geometry can swallow any form of "finitary" logic.

We will think of elements of \mathcal{L} as formulas φ , of course. Now remember that a frame is a pretopos object.

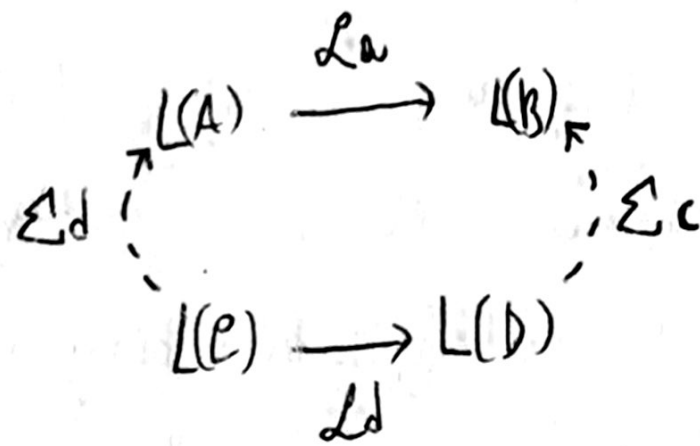
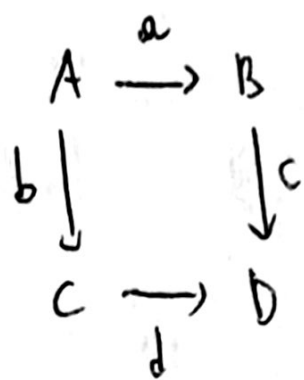
Def A (single sorted) geometric theory is a "dependent frame", that is

$$\mathcal{L} : \text{Fin} \longrightarrow \text{Frames}$$

such that: $\mathcal{L}(I)$ has objects on both sides

$$\mathcal{L}(I) \rightarrow \mathcal{L}(I) \rightarrow \mathbb{T}^{\mathbb{F}}$$

BC For every pb square on the left, the diagram on the right commutes



F

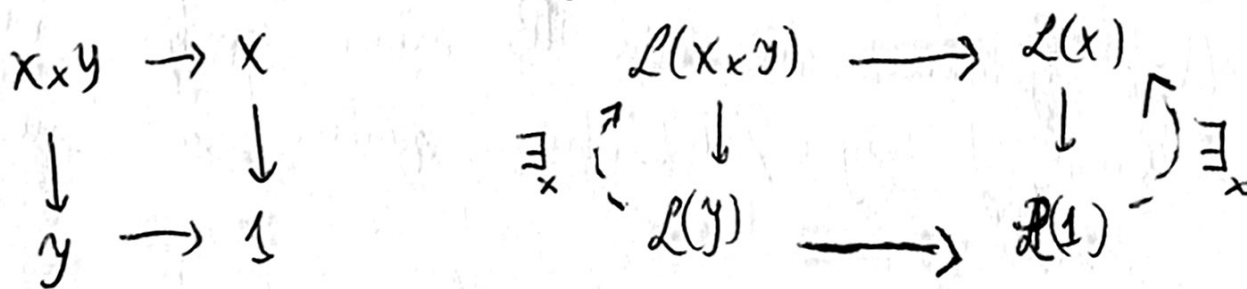
for all $e: A \rightarrow B$, $\varphi \in L(B)$, $\psi \in L(A)$.

this is a form of fibred descent

$$\Sigma(e)(L(e)(\varphi) \wedge \psi) = \varphi \wedge \Sigma(e)(\psi)$$

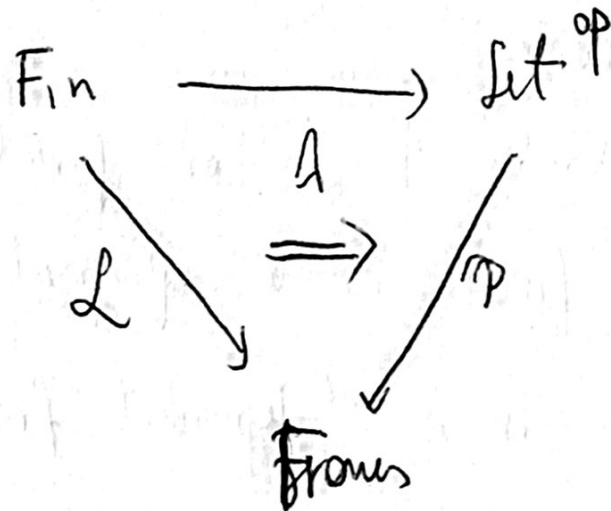
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these correspond to logical rules, for example



Now let us briefly recall the notion of model for such a poset.

A model of L (in Set) is a functor preserving finite coproducts



So that $L(1)$ is the "underlying set" of the model, and of course $L(n) = L(1)^n$ (as a set).

and then

$$\lambda: L(n) \longrightarrow \mathcal{P}(L(n))$$

takes a formula ϕ and interprets it as a formula over $L(n)^n$ (as it should be).

Of course we also want λ to be a morphism of frames.

Now notice that this makes perfect sense when $(\text{Set}, \mathcal{L}^{(-)})$ is substituted with (\mathcal{F}, Ω) . so that we can make sense of models of \mathcal{L} generated by Ω in any topos.

Rem Here it is important that Ω is an internal frame, so that λ makes perfect sense.

Def Let $\mathbb{T}: \text{Fin} \rightarrow \text{Frames}$ be a geometric theory. Then we define

$$\text{Mod}(\mathbb{T})_{(-)}: \text{Topoi} \xrightarrow{\varphi} \text{Cat}$$

$$\mathcal{E} \longmapsto \boxed{\begin{array}{c} \text{Mod}(\mathbb{T}) \\ \mathcal{E} \end{array}}$$

the category of models of \mathcal{L} in \mathcal{E} .

The goal of this lecture is to show that this presheaf is representable, i.e.

$$\text{Mod}(\mathbb{T})_{(-)} \cong \text{Topoi}(-, \boxed{?})$$

this result will allow us to completely absorb geometric logic in the internal logic (i.e. categorical properties) of Topoi .

Part 2 Syntactic Categories.

We have seen that for \mathcal{L} a locale, there is a notion of $\text{Sh}(\mathcal{L})$ that builds a topos out of it. It would be an easy exercise to show that

$$\text{Mod}_{\mathcal{E}}(\mathcal{L}) \cong \text{Topoi}(\mathcal{E}, \text{Sh}(\mathcal{L}))$$

So that $\text{Sh}(\mathcal{L})$ is the class-topos of \mathcal{L} . With this motivation in mind, given a theory

$\Pi: \text{Fin} \rightarrow \text{Frames}$, it would be enough to say "take sheaves over it" to build the class-topos.

Problem What are sheaves for an internal locale?

Answer: There are several answers, and we choose the less elegant one.

We externalize the locale and we make it into a site.

Construction Given a theory $\Pi: \text{Fin} \rightarrow \text{Frames}$, we will define $\text{Syn}\Pi$ to be a site. Now, the best way to collapse a presheaf with its base in such a way that they are mixed is to take the category of elms

$\text{Elt}(\Pi)$

$\text{Elt}_s \Pi \begin{cases} \xrightarrow{\text{obj}} & x_1, \dots, x_n \vdash \varphi \\ \searrow \text{morph} & \text{substitutions} \end{cases} (X \vdash \varphi, (X, \varphi) \bullet)$

Now this guy has a natural topology inherited by the fact that each $\Pi(h)$ is a frame and for those we have a nice notion of covering family. So we obtain

$$\text{Syn} \Pi = (\text{Elt}_s \Pi, \mathcal{J}_{\Pi})$$

Def The clon topos over \mathcal{H} is $\text{Sh}(\text{Elt}_s \Pi, \mathcal{J}_{\Pi})$.

Long discussion on why it works -
 A bit of Dieckmann-Laga

We finish the lecture with two important facts.

[1] Every topos is a classifying topos, and it classifies its theory of flat functors.

[2] $\text{Set}[\mathcal{O}] = \text{Set}^{\text{Fin}}$ is the classifier of the theory of objects which is precisely the one given by its subobject classifier.

$$\Omega \in \text{Set}^{\text{Fin}}$$

$$\mathcal{E} \cong \text{Topos}(\mathcal{E}, \text{Set}[\mathcal{O}])$$

Proof.

[3] For \mathcal{H} a lex theory (not the sense of a category with finite lts) $\text{Mod}(\mathcal{H})_{\omega}$ is the classifier \mathcal{H} .