L15 Classifying topoi.

the plan for the day is the following.

(1) Introduce the notion of geometric theory.
(2) Introduce syntactic categories
(3) Classifying topoi for geometric theories.

(1) Geometric theory.

We have discussed in the lectures on doctrines that one can use "Boolean algebras" (aka fibred posets) to represent theories.

$\perp, \top, \neg, \vee, \wedge, \Rightarrow$ $\quad$ $\mathbb{B}$ $\leftarrow$ boolean algebra.

Of course the algebraic structure $\mathbb{B}$ comes equipped with reflects the kind of logic / deductive system we care about.

<u>Geometric logic is the logic of frames</u>

$\perp, \top, \bigvee, \wedge$ $\quad$ $\mathcal{L}$ $\leftarrow$ frame

In this spirit, a model of a frame is precisely a point, and this is the great ballbreaker that geometry can swallow any form of "finitary" logic.

We will think of elements of $\mathcal{L}$ as formulas φ, of course.

Now remember that a frame is a propositional object.

Def A (single sorted) geometric theory is a "dependent frame", that is

$$\mathcal{L} : \mathrm{Fin} \longrightarrow \mathrm{Frames}$$

such that: $\mathcal{L}(f)$ has adjoints on both sides

$$\Sigma f \dashv \mathcal{L}(f) \dashv \Pi f$$

$\boxed{BC}$ For every pb square on the left, the diagram on the right commutes

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "b"'] & B \arrow[d, "c"] \\ C \arrow[r, "d"'] & D \end{tikzcd} \qquad \begin{tikzcd} \mathcal{L}(A) \arrow[r, "\mathcal{L}a"] & \mathcal{L}(B) \\ \mathcal{L}(C) \arrow[u, dashed, bend left, "\Sigma d"] \arrow[r, "\mathcal{L}d"'] & \mathcal{L}(D) \arrow[u, dashed, bend right, "\Sigma c"'] \end{tikzcd}$$

$\boxed{F}$ for all $a : A \to B$, $\varphi \in \mathcal{L}(B)$, $\psi \in \mathcal{L}(A)$.

this is a form of fibred descent

$$\Sigma(a)\big(L(a)(\varphi) \wedge \psi\big) = \varphi \wedge \Sigma(a)(\psi)$$

$$\varphi \wedge \Sigma a(\psi)$$

these correspond to logical rules, for example

$$\begin{tikzcd} X \times Y \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & 1 \end{tikzcd} \qquad \begin{tikzcd} \mathcal{L}(X\times Y) \arrow[r] \arrow[d] & \mathcal{L}(X) \arrow[d] \\ \mathcal{L}(Y) \arrow[u, dashed, bend left, "\exists_x"] \arrow[r] & \mathcal{L}(1) \arrow[u, dashed, bend right, "\exists_x"'] \end{tikzcd}$$

Now let us briefly recall the notion of model for such a gadget.

A model of $\mathcal{L}$ (in Set) is a functor preserving finite coproducts

$$\begin{tikzcd} \mathrm{Fin} \arrow[rr, "A"] \arrow[dr, "\mathcal{L}"'] & \arrow[d, phantom, "\Longrightarrow"] & \mathrm{Set}^{op} \arrow[dl, "\mathcal{P}"] \\ & \mathrm{Frames} & \end{tikzcd}$$

So that $\mathcal{L}(1)$ is the "underlying set" of the model, and of course $\mathcal{L}(n) = \mathcal{L}(1)^n$ (as a set).

and then

$$\lambda : \mathcal{L}(n) \longrightarrow \mathcal{P}(\mathcal{L}(1)^n)$$

takes a formula φ and interprets it as a formula over $\mathcal{L}(1)^n$ (as it should be).
Of course we also want λ to be a morphism of frames.

Now notice that this makes perfect sense when $(\mathbf{Set}, 2^{(-)})$ is substituted with $(\mathcal{E}, \Omega)$. so that we can make sense of models of a geometric theory in any topos.

Rem Here it is important that Ω is an internal frame, so that λ makes perfect sense.

Def Let $\mathbb{T} : \mathrm{Fin} \to \mathrm{Frames}$ be a geometric theory. then we define

$$\mathrm{Mod}_{(-)}(\mathbb{T}) : \mathrm{Topoi}^{\mathrm{op}} \longrightarrow \mathrm{Cat}$$
$$\mathcal{E} \longmapsto \boxed{\mathrm{Mod}_{\mathcal{E}}(\mathbb{T})}$$

the category of models of $\mathcal{L}$ in $\mathcal{E}$.

The goal of this lecture is to show that this prestack is representable, i.e.

$$\mathrm{Mod}_{(-)}(\mathbb{T}) \simeq \mathrm{Topoi}(-, \boxed{?})$$

this result will allow us to completely absorb geometric logic in the internal logic (i.e. categorical properties) of Topoi.

<u>Part 2</u> Syntactic categories.

We have seen that for $\mathcal{L}$ a locale, there is a notion of $Sh(\mathcal{L})$ that builds a topos out of it. It would be an easy exercise to show that

$$\mathrm{Mod}_{\mathcal{E}}(\mathcal{L}) \simeq \mathrm{Topoi}(\mathcal{E}, Sh(\mathcal{L}))$$

So that $Sh(\mathcal{L})$ is the class-topos of $\mathcal{L}$.
With this motivation in mind, given a theory $\mathbb{T} : \mathrm{Fin} \to \mathrm{Frames}$, it would be enough to say "take sheaves over it" to build the class-topos.

<u>Problem</u> What are sheaves for an internal locale?

Answer: there are several answers, and we choose the less elegant one. We externalize the locale and we make it into a site.

<u>Construction</u> Given a theory $\mathbb{T} : \mathrm{Fin} \to \mathrm{Frames}$, we will define $\mathrm{Syn}\,\mathbb{T}$ to be a <u>site</u>. Now, the best way to collapse a presheaf with its base in such a way that they are mixed is to take the category of elmts

$$\mathrm{Elts}(\mathbb{T})$$

$\mathrm{Elts}\,\mathbb{T}$ objs $\quad x_1, \dots, x_n \vdash \varphi \qquad (X \vdash \varphi, (X,\varphi)\bullet)$

morph $\quad$ substitutions

Now this guy has a natural topology inherited by the fact that each $\mathbb{T}(n)$ is a frame and for those we have a nice notion of covering family. So we obtain

$$\mathrm{Syn}\,\mathbb{T} = (\mathrm{Elts}\,\mathbb{T}, J_{\mathbb{T}}).$$

__Def__ The clone topos over $\mathbb{T}$ is $\mathrm{Sh}(\mathrm{Elts}\,\mathbb{T}, J_{\mathbb{T}})$.

Long discussion on why it works.
A bit of Diaconescu - Yoga

We finish the lecture with two important facts.

[1] Every topos is a classifying topos, and it classifies its theory of flat functors.

[2] $\mathbf{Set}[\mathbb{O}] = \mathbf{Set}^{\mathrm{Fin}}$ is the classifier of the theory of objects ~~which~~ which is precisely the one given by its subobject classifier.

$$\Omega \in \mathbf{Set}^{\mathrm{Fin}}.$$

$$\mathcal{E} \simeq \mathbf{Topoi}(\mathcal{E}, \mathbf{Set}[\mathbb{O}]).$$

Proof.

[3] For $\mathbb{T}$ a lex theory (in the sense of a category with finite limits) $\mathbf{Set}^{\mathrm{Mod}(\mathbb{T})_\omega}$ classifies $\mathbb{T}$.