INTRODUCTION TO CATEGORICAL LOGIC

IVAN DI LIBERTI

rules

- Hand your exercises by the **22nd of March** via email. In order to make my life easier, make sure to include the word **CL23 in the subject**.
- Pick at least one exercise from each of the yellow groups.
- You must charge at least **2** full batteries!
  Example. The vector of exercises [3,7,8,13,16] would pass this sheet.

Exercises

categories

Exercise 1 (). How many idempotent monads $\mathsf{T} : \mathbf{Set} \to \mathbf{Set}$ can you find on the category of sets? Describe them all.

Exercise 2 (). Let $\mathsf{T} : \mathbf{Set} \to \mathbf{Set}$ be a cocontinous monad. Prove or provide a counterexample for the following statement: there exists a monoid M such that $\mathsf{T} \cong \mathsf{M} \times (-)$ as monads.

Exercise 3 (). A *graph* (E, V, s, t) is the data of two sets E, V and two functions $s, t : E \rightrightarrows V$. Morphisms of graphs are defined as expected, and so is the category Gra of graphs. Can you find a full subcategory C containing two objects such that every cocontinuous functor $\mathsf{Gra} \to \mathbf{Set}$ is uniquely determined by its value on C?

Exercise 4 (). Let $\mathcal{A}$ be a cocomplete category and $a \in \mathcal{A}$ be a dense object, i.e. the family consisting of the single object a forms a dense generating set. Show that $\mathcal{A}$ admits a faithful right adjoint $\mathcal{A} \to \mathbf{Set}$ and exhibit a category $\mathcal{A}$ for which it is not an equivalence of categories.

Exercise 5 (,). Let $\mathcal{A}$ be a cocomplete category with a dense generating set. Show that $\mathcal{A}$ is complete.

Exercise 6 (). In the diagram below all the categories are λ-accessible and so are the functors f, g. Assume also that C is cocomplete. Justify that $\mathsf{lan}_g f$ exists and is accessible too.

$$\begin{tikzcd} \mathcal{A} \arrow[r, "f"] \arrow[d, "g"'] & C \\ \mathcal{B} \arrow[ur, dashed, "\mathsf{lan}_g f"'] & \end{tikzcd}$$

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universal algebra

Exercise 7 (,). Consider the categories Grp, Ab of groups and abelian group respectively.

- Describe the Lawvere theories axiomatizing them.
- Show that the inclusion $i : \mathsf{Ab} \hookrightarrow \mathsf{Grp}$ is a morphism of varieties.
- Describe the morphism of Lawvere theories that induce i.

Exercise 8 (,). Consider the categories Grp, Ab of groups and abelian group respectively.

- Describe the **Set** monads axiomatixing them.
- Show that the inclusion $i : \mathsf{Ab} \hookrightarrow \mathsf{Grp}$ is a morphism of varieties.
- Describe the morphism of monads that induce i.

Exercise 9 (,). Recall that the category SLat of suplattices is monadic over $\mathbf{Set}$. Following the standard construction that given a monad produces a (possibly large) algebraic theory, can you describe an equational presentation of the category of suplattices?

Exercise 10 (). Let $\mathsf{T} : \mathbf{Set} \to \mathbf{Set}$ be a finitary monad with some model with two distinct elements, show that its unit is injective.

Exercise 11 (). For every finitary monad $\mathsf{T} : \mathbf{Set} \to \mathbf{Set}$ construct a finitary polynomial monad $\mathsf{P}_{\mathsf{T}} : \mathbf{Set} \to \mathbf{Set}$ and a morphism of monads $\mathsf{P}_{\mathsf{T}} \to \mathsf{T}$ which is pointwise surjective.

sketches

Exercise 12 (). Using the technology of the course show that every abelian group embeds in a divisible one.

Exercise 13 (,). Provide a sketch axiomatizing the category of fields. Could it be a limit sketch?

Exercise 14 (,). Given limit sketches $\mathcal{S}_1, \mathcal{S}_2$ define a symmetric tensor product $\mathcal{S}_1 \otimes \mathcal{S}_2$ in such a way that,

$$\mathsf{Mod}(\mathcal{S}_1 \otimes \mathcal{S}_2, \mathbf{Set}) \simeq \mathsf{Mod}(\mathcal{S}_1, \mathsf{Mod}(\mathcal{S}_2, \mathbf{Set})).$$

Exercise 15 (). Show that the category of Banach spaces and non expansive maps is locally presentable. What about the category of Hilbert spaces?

Exercise 16 (). Show that the category of topological spaces and the category of suplattices are not locally presentable.

Exercise 17 (). Show that if $\mathcal{A}$ is locally finitely presentable, so are $\mathcal{A}^{\to}$ and $\mathcal{A}_{/a}$.

The riddle (Givant, ⚠). A finitary monad $\mathsf{T} : \mathbf{Set} \to \mathbf{Set}$ is *stable* if every algebra is free. (Assuming choice) show that there exactly four families of finitary stable monads $\mathsf{T} : \mathbf{Set} \to \mathbf{Set}$. *Comment.* If you solve it with category theoretic methods, you can publish it.