L10

Type theories as proof relevant hyperdoctrines

1

In the previous lecture we have introduced the notion of doctrine. A doctrine P over a category $\mathcal{C}$ is nothing but an internal poset in its presheaf category

$$P \in Set^{\mathcal{C}^{op}} \quad \& \quad P \text{ is an internal poset.}$$

Of course, not *any* poset. it should at least be a lattice and possibly more.

1.1

When looking at P as an object in $Set^{\mathcal{C}^{op}}$ we gain the Yoneda perspective on the situation. Indeed

$$\alpha \in P(c) \quad \longleftrightarrow \quad \alpha : y(c) \Rightarrow P$$

this means that P as an object "classifies" itself, at the price of embedding $\mathcal{C}$ in its presheaf category

$$Set^{\mathcal{C}^{op}}(y-, P) \simeq P$$

An suddenly P looks like an object *virtually* in $\mathcal{C}$.

this observation helps us to reduce every doctrine to an intuition we are familiar with

1.2

In Set, the object $\underline{2}$ not just virtually, but honestly can be used to understand the subobject doctrine

$Sub : Set^{op} \to Set$	$\nexists 2$ in $Set^{Set^{op}}$	the virtual object exists	$2 \in Set$
$\alpha \in Sub(X)$	$\alpha \; \nexists X \Rightarrow \nexists 2$		$\varphi : X \to 2$

- Ok, so now we are somehow back to the <u>first lecture</u>. (Let's stay in Set for a while)

When we say $X \vdash \varphi$ formula, or, is a subset, or is a proposition, we can classify "facts" about the elements of X.

For X the set of stones on a given land, for example, φ was telling us whether, or not, each step stone has been step on.

- Although, what if we want a device that not only classifies facts, but facts and their evidences? In that case we saw that, for example to store the information of the people that step on a given stone, we could use

"Classifying families over X" | $\varphi : X \to Set$

So now we can do much more

$$\frac{X \vdash \varphi \text{ proposition}}{\int\varphi \underset{\text{mono}}{\hookrightarrow} X} \qquad \Bigg\{ \qquad \frac{X \vdash F \text{ family}}{\int F \xrightarrow{\pi} X}$$

Fair enough, but what kind of categorical tool could we use to organize all of this? What instead of an internal poset in $Set^{C^{op}}$?

- As always the answer is under our nose. (2) is a poset. (Set) is a category. So what we may want is an object

$$\mathcal{U} \in Set^{C^{op}}$$

which is an internal category. For example choose.

$$C = Set$$

$$Set^{Set^{op}} \ni \mathcal{U} = \sharp(\mathcal{U}_\lambda) \qquad \mathcal{U}_\lambda = \text{set of sets of cardinality at most } \lambda.$$

then

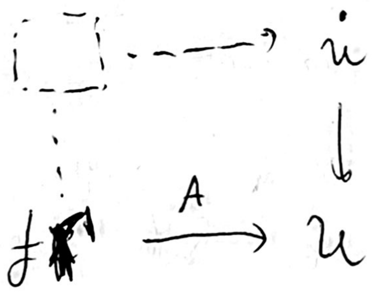
$$\frac{X \vdash F\ \mathcal{U}}{} \rightsquigarrow \frac{F \in \mathcal{U}(X)}{} \rightsquigarrow$$

F is an X-family whose fibers are λ-small.

• Notice now that the category structure (as opposed to the posetal one) is important because for two propositions φ, ψ I can only have $\varphi < \psi$. For two functors I could have ~~many~~ relationships!! $F \rightrightarrows G$.

Natural models of dependent type theory

__Def__ A natural model $p : \dot{\mathcal{U}} \to \mathcal{U}$ over a category $\mathcal{C}$ is a couple of presheaves $\mathcal{U}, \dot{\mathcal{U}}$ together with a natural transformation p between them with the following property. Given a pullback



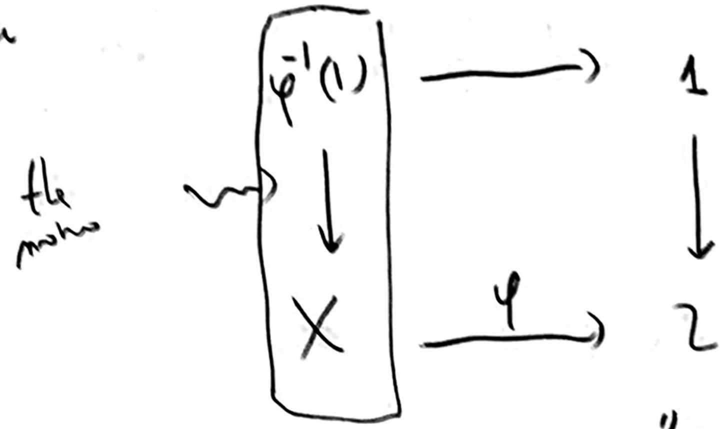
the boxed object is a representable, called $\Gamma . A$.

__Rem__ There are some parts of this definition that we have clearly justified. The idea of using

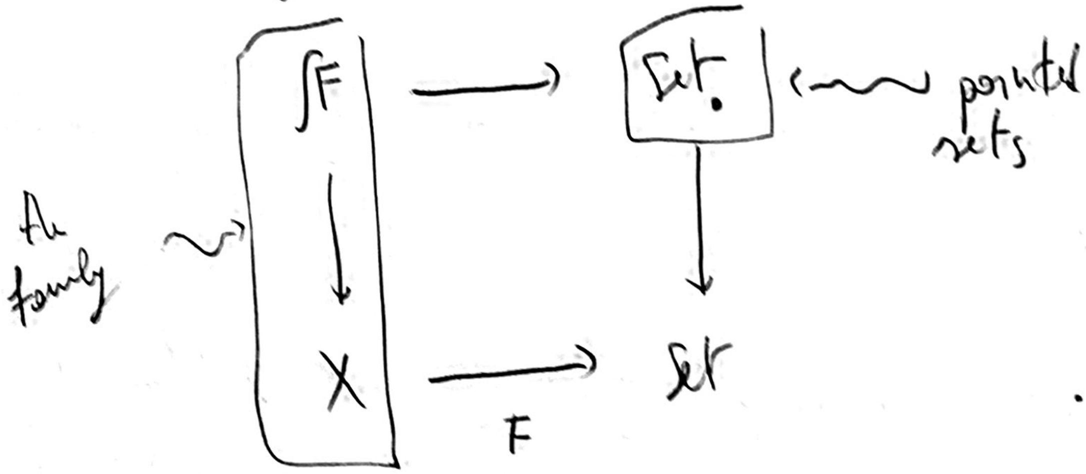
$$y\Gamma \xrightarrow{A} \mathcal{U} \quad \Big\| \quad \Gamma \vdash A \;\; \mathcal{U}$$

but what about this $\dot{\mathcal{U}}$?

Well, the idea again come back from the usual example

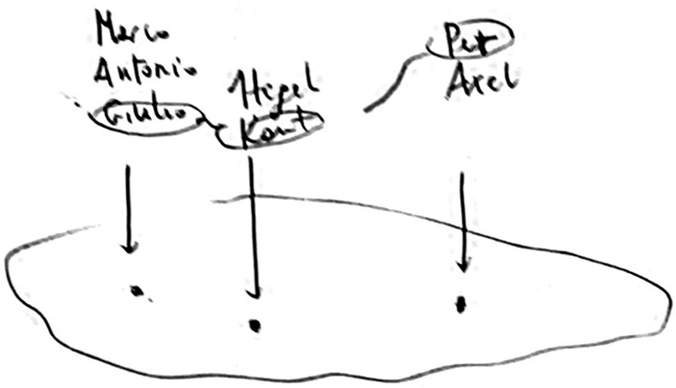


Where $X.\varphi$ is exactly "the subobject" we wanted to specify. Similarly



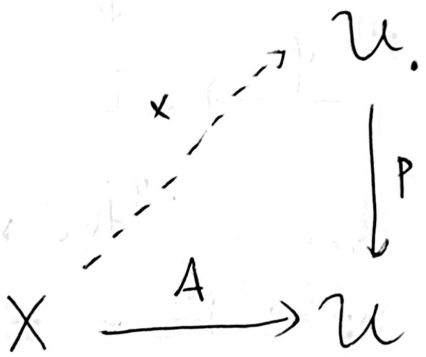
Of course, we do not want our family to be virtual, so we want the pullback to be actually representable.

• Finally, let us notice a "new" addition feature of dependent type theory. We have access to the witnesses!! Consider again the example of the stores



Is there any way, in the model, I can scan the name of a section of my family? Say ~~[illegible]~~ the circled ones

Yes!



An A-section of p is precisely an inhabitant of A!

X ⟼ Steppers Family

X ⟼ ~~[illegible]~~ : Stepper witness
(Galois, Kant, Per)