L12 Topoi as spaces.

Hello. From this lecture and until the end of the course we will talk about topoi. We shall start from topoi "as spaces". This is historically their first appearence.

We will look at topoi from 4 perspectives

- spaces (today).
- sets (next time)
- objects (of the 2-category of topoi).
- theories.

Each topos is, at the same time, a space, a universe of sets, a theory and, of course a guy sitting in a category. The interplay between these fragmented personality makes the richness of this theory.

- The notion of sheaf.

The notion of sheaf comes from topology. But everyone knows that. So, let's go in medias res with an example.

Let X be a topological space and let $\mathcal{O}(X)$ be the poset of its opens.

Then it is of geometric interest to study the assignment

$$\begin{aligned} \mathbb{R} : \mathcal{O}(X)^{op} &\longrightarrow \mathrm{Set} \\ U &\longmapsto \mathrm{Top}(U, \mathbb{R}) \end{aligned}$$

subspace topology.

Originally, the true interest came from the attempt to construct "global elements"

$$f \in \mathbb{R}(X) = \mathrm{Top}(X, \mathbb{R})$$

from the data of some of its restrictions $f_i \in \mathbb{R}(\mathcal{U})$.
Notice that the functor $\mathbb{R}$ has a special property:

The sheaf condition

Let $\mathcal{U}_i$ be a cover of X and let $f_i \in \mathbb{R}(\mathcal{U})$ be a family of continuous functions such that

$$f_i \equiv f_j \text{ on } \mathcal{U}_i \cap \mathcal{U}_j$$

then there exist a globally defined and unique $f \in \mathbb{R}(X)$ s.t $f|_{\mathcal{U}_i} = f_i$

Sheaf theory is the story of presheaves with this property and their heroic actions.

Structure of this lecture

Part 1 [Category of sheaves]

Part 2 [$\mathrm{Et}(X) \simeq \mathrm{Sh}(X)$, a lecture on terminology].

Part 3 [Localic topoi].

Part 1 Category of sheaves.

Def (Sheaf) Let X be a topological space. A sheaf on X is a functor $P : \mathcal{O}(X)^{op} \longrightarrow Set$ with the property that:

$\forall \mathcal{U} \in \mathcal{O}(X)$ and $\mathcal{U}_i : \bigcup \mathcal{U}_i = \mathcal{U}$, given a family $f_i \in P(\mathcal{U}_i)$ such that

$$restr_j f_i = restr_i f_j \in P(\mathcal{U}_i \cap \mathcal{U}_j) \ \forall ij$$

there exist a unique $f \in P(\mathcal{U})$ such that $restr_i (f) = f_i$.

Equivalently

$$P(\mathcal{U}) \xrightarrow{e} \prod_i P(\mathcal{U}_i) \underset{\text{"}k\text{"}}{\overset{\text{"}j\text{"}}{\rightrightarrows}} \prod_{jk} P(\mathcal{U}_j \cap \mathcal{U}_k)$$

e equalizes these two maps.

Def (Category of sheaves) The category of sheaves $Sh(X)$ is the full subcategory of sheaves among presheaves

$$Sh(X) \longrightarrow Set^{\mathcal{O}(X)^{op}}.$$

Of course we get an assignment

$$\begin{aligned} \text{Spaces} &\rightsquigarrow \text{Categories of sheaves} \\ X &\longmapsto Sh(X). \end{aligned}$$

In this **Part 1** we study the properties of the inclusion

$$Sh(X) \hookrightarrow Set^{\mathcal{O}(X)^{op}}$$

Rem 1 There exists a limit sketch structure on $\mathcal{O}(X)^{op}$ such that $Sh(X) \simeq Mod(\mathcal{O}(X)^{op}, \dots)$
This is easy. Indeed in $\mathcal{O}(X)$, we have that the

diagram below has U as a colimit.

$$\begin{tikzcd}
 & U_i \cap U_j \arrow[dl] \arrow[dr] & \\
U_j \arrow[dr] & & U_i \arrow[dl] \\
 & U &
\end{tikzcd}$$

diagram

and thus it is a limit in $\mathcal{O}(X)^{op}$. Then the sheaf condition is exactly telling us that we preserve these limit diagrams.

Rem 2 Recall that via the theory of orthogonality, we obtain that

$$Sh(X) \hookrightarrow Set^{\mathcal{O}(X)^{op}}$$

is an orthogonal class in $Set^{\mathcal{O}(X)^{op}}$ and because this is a small class of diagrams, it is reflective

$$\begin{tikzcd}
Sh(X) \arrow[r, hook, shift right, "i"'] \arrow[r, phantom, "\perp"] & Set^{\mathcal{O}(X)^{op}} \arrow[l, shift right]
\end{tikzcd}$$

the reflector is called sheafification.

Some of you may have heard that the sheafification functor is __lex__ (i.e. preserves finite limits). Let us get convinced of that.

__Rem__ Recall how to transform the sketch condition into an orthogonality problem. Say that I have a colimit diagram D in $\mathcal{C}$, then we have two things to do

$$D \xrightarrow{f} \mathcal{C} \xrightarrow{\text{よ}} \mathrm{Set}^{\mathcal{C}^{op}}$$

From $\operatorname{colim} \text{よ} f$ we get a natural transformation

$$\delta_f : \operatorname{colim} \text{よ} f \Rightarrow \text{よ} \operatorname{colim} f.$$

then we preserve the colimit associated to f iff we are orthogonal to δ_f.

__Rem__ If a reflection is lex, then the class of maps inverted by the reflector is pullback stable

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d, "l"'] \arrow[dr, phantom, "\lrcorner", very near start] & \cdot \arrow[d, "\mathcal{H}"] \\
\cdot \arrow[r] & \cdot
\end{tikzcd}
\qquad
\begin{tikzcd}
L \arrow[r] \arrow[d, "\text{must be iso}"'] \arrow[dr, phantom, "\lrcorner", very near start] & L \arrow[d, "L(\mathcal{H})"] \arrow[d, phantom, "\text{is} \cong"', shift right=4] \\
L \arrow[r] & L
\end{tikzcd}$$

$\Rightarrow l \in \mathcal{H}$ too.

__thm__ (Borceux) the converse is true. If $\mathcal{H}$ is a p.b.-stable class, then $\mathcal{H}^{\perp}$ is reflective and the reflector is lex.

So now we show that our class is p.b stable. If we open up the definitions, it reduces to

$$\begin{tikzcd}
 & U_i\cap U_j\cap V \arrow[dl] \arrow[dr] \arrow[rrr, dashed] & & & U_i\cap U_j \arrow[dl] \arrow[dr] & \\
U_i\cap V \arrow[dr] \arrow[rrr, dashed, bend left] & & U_j\cap V \arrow[dl] \arrow[rrr, dashed] & U_i \arrow[dr] & & U_j \arrow[dl] \\
 & V \arrow[rrr] & & & U &
\end{tikzcd}$$

And of course this p.b. family is still in our class, because

$$V = \left[\bigcup U_i\right] \cap V = \bigcup \left[U_i \cap V\right].$$

descent... Infinitary distributivity rule.

So, ok. we have that $Sh(X) \underset{\hookrightarrow}{\leftarrow} Psh(\mathcal{O}(X))$ and the reflector is lex.

Locales vs Spaces

Notice that what was important about $\mathcal{O}(X)$ was that

- It is a poset with $\bigvee$ and $\wedge$.
- we have the infinitary dist rule.

Def A frame is that. Of course we automatically get the notion of sheaf over a locale $\mathcal{L}$ and

$$Sh(\mathcal{L}) \underset{\hookrightarrow}{\leftarrow} Psh(\mathcal{L}) \text{ is lex-reflective in } Psh(\mathcal{L}).$$

Locales are "kinda" spaces.

Standard presentation of the adjunction

$$\begin{tikzcd} \mathrm{Loc} \arrow[r, bend right, "\mathcal{O}"'] \arrow[r, phantom, "\top"] & \mathrm{Top} \arrow[l, bend right, "\mathrm{pt}"'] \end{tikzcd}$$

Check any book on the topic.

Part 2 A lecture in terminology.

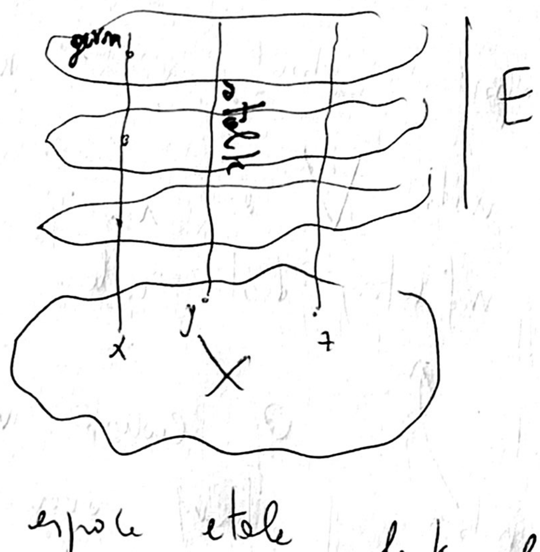
So why do we call a "sheaf" in this way? What is the geometric intuition behind this notion? Why my friends talk about "stalks" and "germs"?

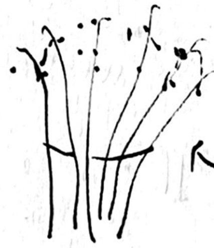
Def Let X be a topological space. An espace etale $E \xrightarrow{p} X$ is a local homeomorphism.

Examples $=$ • $\mathcal{U} \hookrightarrow X$ (opens).

• covering maps $E \xrightarrow{p} \cancel{X}$.

Picture





$\rightsquigarrow$ sheaf.

So, an espace etale looks like a "sheaf".

A morphism of etale spaces is a deck transformation

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "p_2"'] & & E \arrow[dl, "p_1"] \\ & X & \end{tikzcd}$$

this gives us a category $Et(X)$.

<u>thm</u> $Et(X) \simeq Sh(X)$.

<u>Proof</u> If the topology is discrete, this is the grothendieck construction !!

$$Set/_X \simeq Set^X.$$

then we follow this intuition...

<u>Part 3</u> Localic Topoi

At some point of these lectures topoi will be objects of a category. So, we better introduce a notion of morphism between them.

<u>Motivation</u> Let $f : X \to Y$ be a continuous function, and let $f^* : \mathcal{O}(Y) \to \mathcal{O}(X)$ be the induced morphism of frames.
It is clear that if

$$P : \mathcal{O}(X)^{op} \to Set \text{ is a sheaf, so is the}$$

composition

$$\mathcal{O}(Y)^{op} \xrightarrow{f^{*op}} \mathcal{O}(X)^{op} \longrightarrow Set$$

because f^* preserve the covering structure.

So we get...

$$\begin{tikzcd}
\mathrm{Sh}(X) \arrow[rr] \arrow[d, shift left] & & \mathrm{Sh}(Y) \arrow[d] \\
\mathcal{O}(X)^{op} \arrow[u, shift left] & \boxed{\mathrm{lex}} & \mathcal{O}(Y)^{op} \arrow[ll, "f"'] \arrow[u, shift right] \\
\mathrm{Set} \arrow[rr, shift left] \arrow[uu, bend left, "\mathrm{lex}"] \arrow[rr, phantom, "\perp"] & & \mathrm{Set} \arrow[ll, shift left, bend left]
\end{tikzcd}$$

One can check by abstract nonsense that the left object is the real composition which because it is a composition of lex functors it will be lex.

So a "geometric morphism" is an adjunction

$$\begin{tikzcd}
\mathrm{Sh}(\mathcal{G}) \arrow[r, shift right, "f_*"'] \arrow[r, phantom, "\perp"] & \mathrm{Sh}(\mathcal{L}) \arrow[l, shift right, "f^*"']
\end{tikzcd}$$

such that f^* is lex.

$$\text{Locales} \xrightarrow{\mathrm{Sh}} \text{Localic topoi}$$

$\boxed{\text{Question}}$ Given a localic topos, is the information about its locale hidden anywhere?

Rem Of course we have

$$\mathcal{L} \xrightarrow{y} Sh(\mathcal{L})$$

and because y preserves limits, we have

$$\begin{tikzcd}
\mathcal{L} \arrow[rr, "y"] \arrow[dr] & & Sh(\mathcal{L}) \\
& Sub(1) \arrow[ur] &
\end{tikzcd}$$

Prop There is an equivalence of categories

$$\mathcal{L} \simeq Sub(1)$$

Proof For $P \rightarrowtail 1$ a subobject, we define

$$V = \sup_{l \in \mathcal{L}} P(l) = \{1\} .$$