Kan extensions

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"] \arrow[dd, bend right=60, "\text{よ}"'] & C \\
B \arrow[d, "g^{-1}"] & \\
\mathcal{P}(A) \arrow[uur] &
\end{tikzcd}$$

C ← cocomplete

$$\mathrm{Lan}_{\text{よ}}(f)(a) = \text{"}\sup_{x \in g^{-1}(a)} f(x)\text{"}.$$

Problem! If C is cocomplete but A is large, this is cheating.

Solution

$$\mathcal{P}(A) \hookrightarrow \mathrm{Set}^{A^{op}}$$

↖ small presheaves

$B(f-, -)$ is small.

Define arity!!
Ex Yoneda

Prop If g has arity $A' \subset A$, then the extension exist

$$\mathrm{Lan}_{gj} f.j = \mathrm{Lan}_{g}(\mathrm{Lan}_{j} f_j) = \mathrm{Lan}_{j}(f).$$

__AFT__ $\quad A \xrightarrow{f} B$

Both cocomplete. f preserve colimits.

then $\quad r = \mathrm{Lan}_f 1 = \mathrm{Lan}_{\cancel{f}} \cancel{1} \circ \boxed{\mathrm{Lan}_f f}$

Solution set condition

$B(f-, -)$.

~~"Reflectivity"~~

$\cancel{\underset{\text{complete}}{A} \xrightarrow{j} B \xrightarrow{\ \ } \underset{\text{cocomplete}}{C}}$

Lawvere theories ("Highlights")

(1) Axiomatizing Mon

- Lots of implicit operations.
- $\mathcal{U} : \mathrm{Mon} \longrightarrow \mathrm{Set}$
- $\mathrm{Nat}(\mathcal{U}^n, \mathcal{U}^m)$.

(*) $\mathrm{Nat}(\mathcal{U}^n, \mathcal{U}^m) \cong \mathrm{Mon}(F_m, F_n)$.

(2)

$$\mathbb{T} \begin{cases} \longrightarrow \text{numbers} \\ \longrightarrow \mathbb{T}(n, m) = \mathrm{Nat}(\mathcal{U}^n, \mathcal{U}^m). \end{cases}$$

- (*) $\Rightarrow \mathbb{T}$ has products.
- Equality between compositions axiomatize equations

$$\begin{cases} \text{operations} \\ \text{Equations.} \end{cases}$$

(3) Functorial Semantics

- $\mathrm{Prod}(\mathbb{T}, \mathrm{Set}) \longrightarrow \mathrm{Mon}$

defined in this very special case via

$F(1)$

$\mu : F(2) \longrightarrow F(1).$

$F(2) = F(1) \times F(1)$

$\boxed{\text{What about the other direction?}}$

- Well $\mathbb{T}^{op} \longrightarrow \mathrm{Mon}$, then nerve.
- Multisorted !!

(4) Properties of $\mathrm{Mod}(\mathbb{T})$.

- $\mathrm{Prod}(\mathbb{T}, \mathrm{Set}) \hookrightarrow \mathrm{Set}^{\mathbb{T}}$ closed under limits.

ARV Mention What are sifted colimits?

- Has a dense generator.

"sifted colimits".

- it is closed under $\underbrace{\text{directed colimits \& reflexive coeq.}}$

Def dir colim / filtered colimit.

Def ref coeq

- Mention Barr-Beck

$$\boxed{\begin{smallmatrix}\longrightarrow\\ \longleftarrow\\ \longrightarrow\end{smallmatrix}}$$

- $D \to D^2$ is (co)final.
- (Prop Categories with finite products are cosifted).
- Commutation of limits and colimits.
- Representables are "tiny".
- It is complete $\rightsquigarrow$ intro to the s.o.a

$$\begin{tikzcd}
f^{\#}(a \times b) \arrow[r] \arrow[d] & F \\
f^{\#}a \times f^{\#}b \arrow[ur, dotted] &
\end{tikzcd}$$

(5) "Interpretation of theories".