L3 Highlights

Varieties
a syntax semantics duality.

(1) We saw that a category with products $\mathcal{C}$ can be seen as an "operational" theory, and this is even more evident when we specify a b.o. functor $\mathrm{Fin}^{op}_{/X} \xrightarrow{\ell} \mathcal{C}$ preserving products. Indeed,

$$\mathrm{Mod}(\mathcal{C}) \xrightarrow{\ell^*} \mathrm{Prod}(\mathrm{Fin}^{op}_{/X}, \mathrm{Set}) \simeq \mathrm{Set}^X$$

the functor ℓ^* gives us a forgetful functor that declares a family of relevant sorts for the "theory" axiomatized by $\mathcal{C}$.

(2) We saw that the inclusion

$$\begin{tikzcd} \mathrm{Mod}(\mathcal{C}) \arrow[r, hook, "i"] \arrow[r, phantom, "\perp", shift left=2] & \mathrm{Set}^{\mathcal{C}} \arrow[l, dashed, bend right, "L"'] \end{tikzcd}$$

- creates limits (trivially).
- creates directed & sifted colimits (they commute with products in Set).
- has a left adjoint (only sketched, postponed to orthogonality theory).
- has a dense subcategory $\mathcal{C}^{op} \xrightarrow{\sharp} \mathrm{Mod}(\mathcal{C})$.

Q1: $\boxed{\text{But}}$ can we characterize "algebraic categories"?

Q2: Do the properties we listed characterize them?

So, the honest answer to Q1 is yes but in many equivalent ways. In this lecture we do not choose the original one provided by Lawvere, nor the most essential one, which emerged in a long process of maturation involving several people (Pedicchio, Linton, Vitale, Adámek, Rosický, ...). We choose the one that most easily generalises to other contexts.

Let us do the first steps in the direction of answering the question.

Remark. In general the objects of the form $\text{よ}c$

$$\text{よ} : \mathcal{C} \longrightarrow \mathrm{Psh}(\mathcal{C})$$

are tiny, i.e. the hom functor preserve all colimits, indeed

$$
\begin{aligned}
\mathrm{Psh}(\mathcal{C})(\text{よ}c, \operatorname{colim} P_i) &\overset{\text{Yoneda}}{\cong} (\operatorname{colim} P_i)(c) \\
&\overset{\text{colimits are pointwise}}{\cong} \operatorname{colim} \left[P_i(c) \right] \\
&\overset{\text{Yoneda}}{\cong} \operatorname{colim} \mathrm{Psh}(\mathcal{C})(\text{よ}c, P_i). \qquad \square
\end{aligned}
$$

Now, because the inclusion creates sifted colimits,

$$\mathrm{Mod}(\mathcal{C}) \xrightarrow[i]{} \mathrm{Set}^{\mathcal{C}}$$

it follows that the representable models are "tiny" with respect to those.

$$\mathrm{Mod}(\mathcal{C})(\sharp c, \operatorname{colim} x_i) \overset{\text{fullness}}{\cong} \mathrm{Set}^{\mathcal{C}}(\sharp c, i(\operatorname{colim} x_i))$$

i creates sifted colim $\cong \mathrm{Set}^{\mathcal{C}}(\sharp c, \operatorname{colim} i x_i)$

$\cong \ldots$ ☐.

$\boxed{\text{sifted} = \text{directed} + \text{reflexive coeq}}$

Def Finitely presentable object

Examples : ... finite stuff.

Reflexive coequalizers & projectivity

Examples : free algebras.

So, in a "variety" whatever this should be we have dense generator made of these

$\boxed{\text{finite, projective objects.}}$

Prop the full subcategory of these objects is closed under coproducts.

Def A (multisorted) **variety** is a cocomplete category with a dense generator made of "finite projective" objects.

Morphisms of varieties.

- Fin^{op}, after all, is a "theory" itself, and its models are sets.

$$\mathrm{Mod}(\mathrm{Fin}^{op}) \simeq \mathrm{Set}.$$

the functor specifying the sorts of a theory $\ell : \mathrm{Fin}^{op} \to \mathcal{C}$ induces a functor

$$\mathrm{Mod}(\mathcal{C}) \xrightarrow{\ell^*} \mathrm{Mod}(\mathrm{Fin}^{op})$$

- preserving limits
- preserving sifted colimits. (AFT ⟹ has a left adjoint!!).

__Prop__ the left adjoint preserve finite projectives.

__Df__ A morphism of varieties is a right adjoint preserving sifted colimits.

__Ex__ Set^{C} is a variety and also the inclusion

$$\mathrm{Mod}(C) \longrightarrow \mathrm{Set}^{C}$$ is

a morphism of varieties!

__Rem__ Interpretations ...

Duality

$$\begin{tikzcd} \mathrm{Prod} \arrow[rr, bend left, "\mathrm{Mod}"] & & \mathrm{Ver} \end{tikzcd}$$

- Mod was essentially already defined.
- In the other direction we map V to the full subcategory of "finite projectives" V_ω^P

Rem (Makkai–Lawvere) $\mathrm{Mod} \simeq \mathrm{Prod}(-, \mathrm{Set})$

$$V_\omega^P \simeq \mathrm{Ver}(-, \mathrm{Set})$$

Set is the dualizing object.