L4 Monads

This lecture will be about <u>finitory monads</u>. But we need a bit of preparation.

- In the previous lecture we presented two constructions

$$\begin{tikzcd} \mathrm{Prod}^{op} \arrow[rr, bend left, "\mathrm{Mod}"] \arrow[rr, phantom, "\perp"] & & \mathrm{Var} \arrow[ll, bend left, "(-)^{op}_{wp}"] \end{tikzcd}$$

Now we shall discuss unit and counit.

(η) Let $\mathcal{W}$ be a variety and consider the diagram below

$$\begin{tikzcd} & \mathcal{W}_{wp} \arrow[dl, "i"'] \arrow[dr, "j"] & \\ \mathcal{W} & & \boxed{\mathrm{Prod}(\mathcal{W}^{op}_{wp}, \mathrm{Set})} \simeq \mathrm{Mod}(\mathcal{W}^{op}_{wp}) \end{tikzcd}$$

The nerve functor $N(i) : \mathcal{W} \longrightarrow \mathrm{Psh}(\mathcal{W}_{wp})$ lands in $\mathrm{Mod}(\mathcal{W}^{op}_{wp})$, because $\mathcal{W}_{wp}$ is closed under coproducts. Indeed

$N(i)(x) = \mathcal{W}(i-, x)$, but i preserve finite coproduct and thus the composition preserve finite products (Mind the op!).

$$\mathcal{W} \xrightarrow{N(i)} \mathrm{Mod}(\mathcal{W}^{op}_{wp})$$

the nerve is clearly continuous, and preserve sifted colimits. So, it is a morphism of varieties.

(E) Let $\mathcal{C}$ be a category with products. Then the Yoneda embedding

$$\mathcal{C}^{op} \xrightarrow{\text{よ}} \mathrm{Mod}(\mathcal{C})$$

lands in the full subcategory of sifted-tiny objects and preserve finite coproducts.

$$\begin{tikzcd}
\mathcal{C}^{op} \arrow[rr] \arrow[dr] & & \mathrm{Mod}(\mathcal{C}) \\
& \mathrm{Mod}(\mathcal{C})_{wp} \arrow[ur] &
\end{tikzcd}$$

So we get a map preserving products.

$$\mathcal{C} \longrightarrow \mathrm{Mod}(\mathcal{C})_{wp}^{op}$$

Which <u>in the opposite</u> of Prod is a counit.

<u>thm</u>

$$\boxed{\begin{array}{l} \eta \text{ is an equivalence of categories.} \\ \mathcal{C} \text{ is "up to Cauchy completion".} \end{array}}$$

So, Var is coreflective in Prod^{op}.

- <u>Dualizing objects</u>

Dualities are based on dualizing objects.

$$\begin{tikzcd} A^{op} \arrow[rr, bend left, "A(-,a)"] & & B \arrow[ll, bend left, "B(-,b)"] \end{tikzcd}$$

And often the dualizing object is "somewhat the same" object inhabiting both categories.
The functor $\mathrm{Mod} \simeq \mathrm{Prod}(-, \mathrm{Set})$ is representable, so there is hope for an honest duality

<u>Prop</u>

$$W_{wp} \simeq \mathrm{Par}(W, \mathrm{Set})$$

<u>Proof</u>

$$\begin{aligned} \mathrm{Par}(W, \mathrm{Set}) &\simeq \mathrm{RAdj}_{sift}(W, \mathrm{Set}) \\ &\simeq \mathrm{LAdj}_{sift\text{-}tiny}(\mathrm{Set}, W)^{op} \\ &\simeq \left\{ \begin{array}{l} \text{functors } 1 \to W \text{ whose image} \\ \text{is sifted-tiny} \end{array} \right\}^{op} \\ &\simeq W_{wp}^{op}. \end{aligned}$$

- <u>Sorting</u>

Our duality is not sorted at the moment, in the sense that we lost track of the functor

$$\mathrm{Fin}^{op} \longrightarrow \mathcal{L}$$

specifying the theory. But this is easy to recover, at least a portion of it.

When there is an adjunction,

$$\begin{tikzcd} A \arrow[r, bend left, "R"] \arrow[r, phantom, "\top"] & B \arrow[l, bend left, "\mathcal{L}"] \end{tikzcd}$$

We can slice and get a new adjunction

$$\begin{tikzcd} A/a \arrow[r, bend left, "R"] \arrow[r, phantom, "\top"] & B/Ra \arrow[l, bend left, "\mathcal{L}"] \end{tikzcd}$$

Let's do it in our case

$$\begin{tikzcd} \mathrm{Prod}^{op}/\mathrm{Fin} \arrow[r, bend left] & \mathrm{Var}/\mathrm{Mod}(\mathrm{Fin}) \arrow[l, bend left] \end{tikzcd}$$

- An object here is a functor preserving products

$$\mathrm{Fin} \longrightarrow \mathcal{L}$$

- A morphism is

$$\begin{tikzcd} & \mathrm{Fin} \arrow[dl] \arrow[dr] & \\ \mathcal{L} \arrow[rr, "f"] & & T \end{tikzcd}$$

- An object here is a variety with a morphism of varieties

$$V \longrightarrow \mathrm{Set}$$

- A morphism is

$$\begin{tikzcd} V \arrow[rr] \arrow[dr] & & W \arrow[dl] \\ & \mathrm{Set} & \end{tikzcd}$$

So, we almost recovered the notion of Law theory. The problem is being bijective on objects.

Q: What kind of functors correspond to b.o. functors $Fin \longrightarrow \mathcal{L}$? They must be some special right adjoints preserving sifted colimits

$$\mathcal{V} \xrightarrow[f_*]{} Set$$

$\boxed{\text{Monads}}$

We start with a proposition.

Prop Let $f : Fin^{op} \longrightarrow \mathcal{L}$ be a theory. Then the functor

$$Mod(\mathcal{L}) \xrightarrow{f^*} Set$$

is monadic.

Proof Faithfulness follows from the observation that when f is b.o., the bottom functor is monadic.

$$\begin{tikzcd}
Mod(\mathcal{L}) \arrow[r, "f^*"] \arrow[d] & Set \arrow[d] \\
Set^{\mathcal{L}} \arrow[r, "f^*"'] & Set^{Fin}
\end{tikzcd}$$

conservative

& the top is faithful by easy inspection.

Monadicity then follows by Beck. Indeed

f^* is (1) a right adj.

(2) is conservative

(3) creates reflexive coequalizers!

<u>this comes for free because we preserve sifted colimits!!</u>

- So if f is bo the functor

$$\mathcal{W} \xrightarrow{f_*} \mathrm{Set} \quad \text{is monadic.}$$

In the other direction, if we have a monadic functor

$$\mathcal{W} \xrightarrow{f_*} \mathrm{Set}, \quad \text{then}$$

the induced map $\mathrm{Fin} \xrightarrow{f^*} \mathcal{W}_{up}$ is easily <u>bo</u>.

fully faithfull because $\mathcal{W} = \mathrm{Alg}(f_* f^*)$

$$\begin{tikzcd}
\mathrm{Free}\, f^* \arrow[rr] & & \mathcal{W} \arrow[dl, shift right, "f_*"'] \\
& \mathrm{Set} \arrow[ul, "\mathrm{Free}"] \arrow[ur, shift right, "f^*"'] & \\
& \mathrm{Fin} \arrow[u, hook] &
\end{tikzcd}$$

bo because $\mathrm{Free} : \mathcal{C} \to \mathrm{Kl}(T)$ is always bo

So the corestriction

$$\mathrm{Fin} \longrightarrow \mathcal{W}_{up} \quad \text{is bo.}$$

$$\text{Law} \rightleftarrows \begin{array}{l}\text{Monadic morphisms} \\ \text{of varieties} \longrightarrow \mathrm{Set}\end{array}$$

this is the pic we got.

Of course, a monadic morphism $\mathcal{V} \xrightarrow{f} \mathrm{Set}$ such that f_* preserve sifted colimits is the same as a monad on Set preserving sifted colimits

$$\text{Law} \qquad\qquad\qquad\qquad \mathrm{Mnd}(\mathrm{Set})_{\mathrm{sift}}$$

$$\boxed{\mathrm{Fin}^{op} \longrightarrow Kl(T)^{op}} \qquad\qquad T : \mathrm{Set} \longrightarrow \mathrm{Set}$$

$$\mathrm{Fin}^{op} \hookrightarrow \mathrm{Set}^{op} \longrightarrow Kl(T)^{op}$$

$$F : \mathrm{Set} \to Kl(T)$$

$$\mathrm{Fin}^{op} \longrightarrow \mathcal{L} \qquad\qquad \boxed{f_* f^* : \mathrm{Set} \to \mathrm{Set}}$$

$$\mathrm{Mod}(\mathcal{L}) \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathrm{Set}$$

Prop On the category of sets, a monad preserves sifted colimits iff it preserves directed colimits.

$$\mathrm{Mnd}_{filt}(Set) \simeq \mathrm{Mnd}_{sift}(Set)$$

Morphisms of Monads

So we discussed a correspondence between

$$\begin{tikzcd} \mathrm{Law} \arrow[r, dashed, bend left] & \mathrm{Mnd}_{sift}(Set) \arrow[l, dashed, bend left] \end{tikzcd}$$

But what about the morphisms ?!

For Law theories we have natural transformations.

For monads ?

Def A morphism of monads $T \overset{\varphi}{\Longrightarrow} S$ is a natural transformation which is a morphism of monoids

Easy to see that morphism of theories go to morphisms of monads