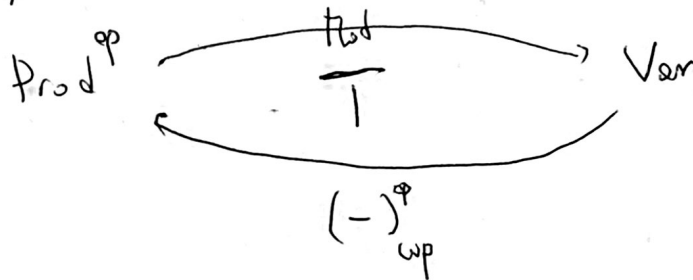


L4 Monads

Highlights

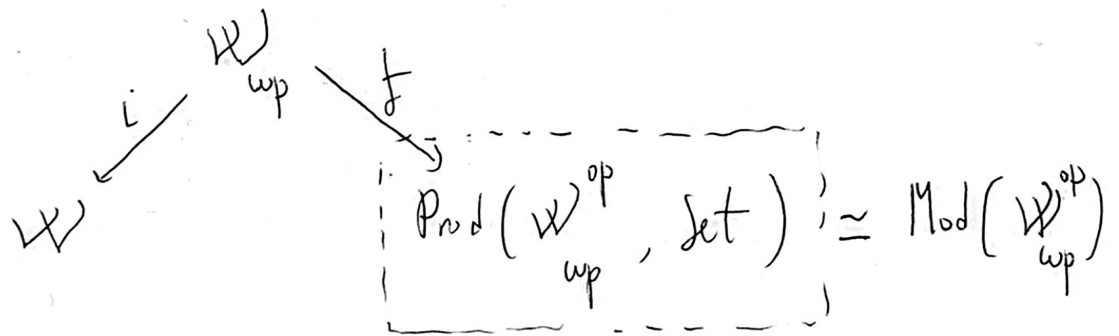
this lecture will be about finite monads. But we need a bit of preparation.

- In the previous lecture we presented two constructions



Now we shall discuss unit and counit.

(*) Let \mathcal{W} be a variety and consider the diagram below



The nerve functor $N(i): \mathcal{W} \rightarrow \text{Psh}(W_{wp}^{\text{op}})$ lands in $\text{Mod}(W_{wp}^{\text{op}})$, because W_{wp}^{op} is closed under coproducts. Indeed

$N(i)(x) = \mathcal{W}(i-, x)$, but i preserve finite coproduct and thus the composition preserve finite products (Mind the $\text{op}!$).

$$\mathcal{W} \xrightarrow{N(i)} \text{Mod}(W_{wp}^{\text{op}})$$

The nerve is clearly continuous, and preserve sifted colimits. So it is a morphism of varieties.

(E) Let \mathcal{C} be a category with products. Then the Yoneda embedding

$$\mathcal{C}^{\text{op}} \xrightarrow{\neq} \text{Mod}(\mathcal{C})$$

lands in the full subcategory of sifted-finite objects and preserve finite coproducts.

$$\begin{array}{ccc} \mathcal{C}^{\text{op}} & \xrightarrow{\quad} & \text{Mod}(\mathcal{C}) \\ & \searrow & \nearrow \\ & \text{Mod}(\mathcal{C})_{\text{up}} & \end{array}$$

So we get a map preserving products.

$$\mathcal{C}^{\text{op}} \xrightarrow{\quad} \text{Mod}(\mathcal{C})_{\text{up}}^{\text{op}}$$

Which in the opposite of Prod is a comt.

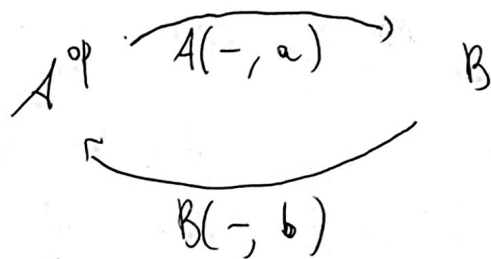
Thm

η is an equivalence of categories.
 \mathcal{E} is "up to Kanby completion".

So, Var is coreflective in Prod^{op} .

• Duality objects

Dualities are based on duality objects.



And often the duality object is "somewhat the same" object inhabiting both categories.

The functor $\text{Mod} \cong \text{Prod}(-, \text{Set})$ is representable, so there is hope for an honest duality.

Prop

$$\mathcal{W}_{\text{op}} \cong \text{Ver}(\mathcal{W}, \text{Set})$$

Proof

$$\text{Ver}(\mathcal{W}, \text{Set}) \cong \text{RA}_{\text{sift}}(\mathcal{W}, \text{Set})$$

$$\cong \text{LA}_{\text{sift-ting}}(\text{Set}, \mathcal{W})^{\text{op}}$$

$$\cong \left\{ \text{functors } 1 \rightarrow \mathcal{W} \text{ whose image is sifted-ting} \right\}^{\text{op}}$$

$$\cong \mathcal{W}_{\text{op}}$$

• Sorting

Our duality is not sorted at the moment, in the sense that we lost track of the functor

$$\text{Fin}^{\text{op}} \xrightarrow{\quad} \mathcal{L}$$

specifying the theory. But this is easy to recover, at least a portion of it.

When there is an adjunction,

$$A \begin{array}{c} \xrightarrow{R} \\ T \\ \xleftarrow{L} \end{array} B$$

We can slice and get a new adjunction

$$A/a \begin{array}{c} \xrightarrow{R} \\ T \\ \xleftarrow{L} \end{array} B/Ra$$

Let's do it in our case

$$\text{Prod}^{\text{op}} / \text{Fin} \begin{array}{c} \xrightarrow{\quad} \\ \quad \\ \xleftarrow{\quad} \end{array} \text{Var} / \text{Mod}(\text{Fin})$$

• An object here is a functor preserving products

$$\text{Fin} \longrightarrow \mathcal{L}$$

• A morphism is

$$\begin{array}{ccc} & \text{Fin} & \\ & \swarrow \quad \searrow & \\ \mathcal{L} & \xrightarrow{F} & T \end{array}$$

• An object here is a variety with a morphism of varieties

$$V \longrightarrow \text{Set}$$

• A morphism is

$$\begin{array}{ccc} V & \longrightarrow & W \\ & \searrow \quad \swarrow & \\ & \text{Set} & \end{array}$$

So, we almost recovered the notion of Law theory. The problem is being bijective on objects.

Q: What kind of functors correspond to law functors $\text{Fin} \rightarrow \mathcal{L}$? They must be some special right adjoints preserving sifted colimits

$$\mathcal{V} \xrightarrow{\quad} \text{Set}$$

\downarrow
 f
 $*$

Methods

We start with a proposition.

Prop Let $f: \text{Fin}^{\text{op}} \rightarrow \mathcal{L}$ be a theory. Then the functor

$$\text{Mod}(\mathcal{L}) \xrightarrow{f^*} \text{Set}$$

is monadic.

Proof Faithfulness follows from the observation that when f is ~~not~~ law, the bottom functor is monadic.

$$\begin{array}{ccc} \text{Mod}(\mathcal{L}) & \xrightarrow{f^*} & \text{Set} \\ \downarrow & & \downarrow \\ \text{Set } \mathcal{L} & \xrightarrow{f^*} & \text{Set}^{\text{Fin}} \end{array}$$

A commutative

So the top ~~one~~ is faithful by easy inspection. (5)

Monadicity then follows by Beck. Indeed

- f^* is
- (1) a right adj.
 - (2) is conservative
 - (3) creates reflexive coequalizers!

This comes for free because we preserve sifted colimits !!

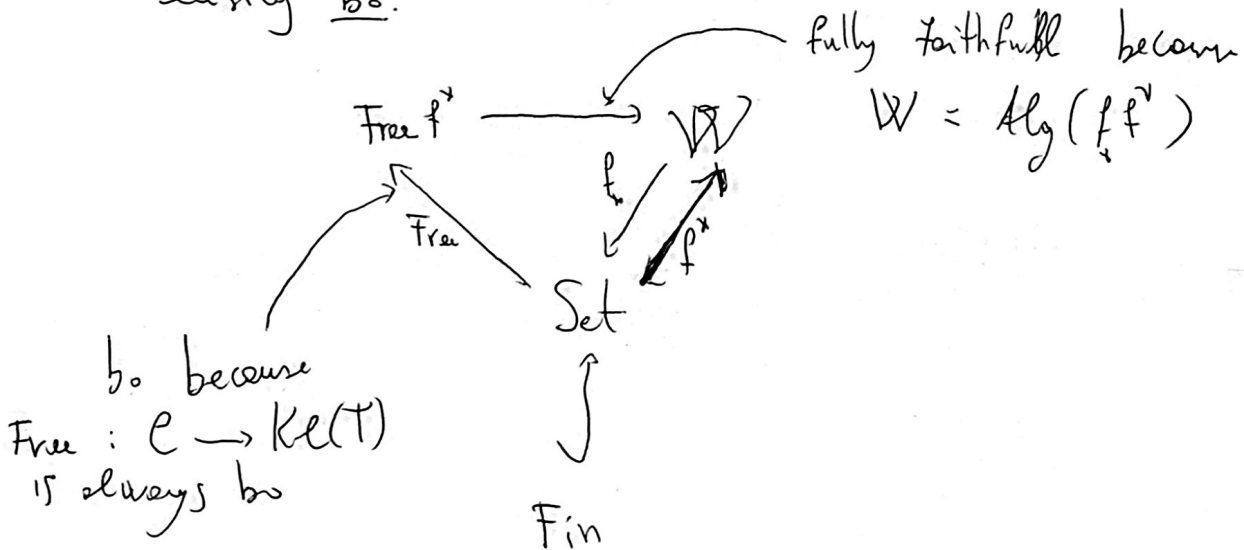
• So if f is bo the functor

$$\mathcal{W} \xrightarrow{f_*} \text{Set} \text{ is monadic.}$$

In the other direction, if we have a monadic functor

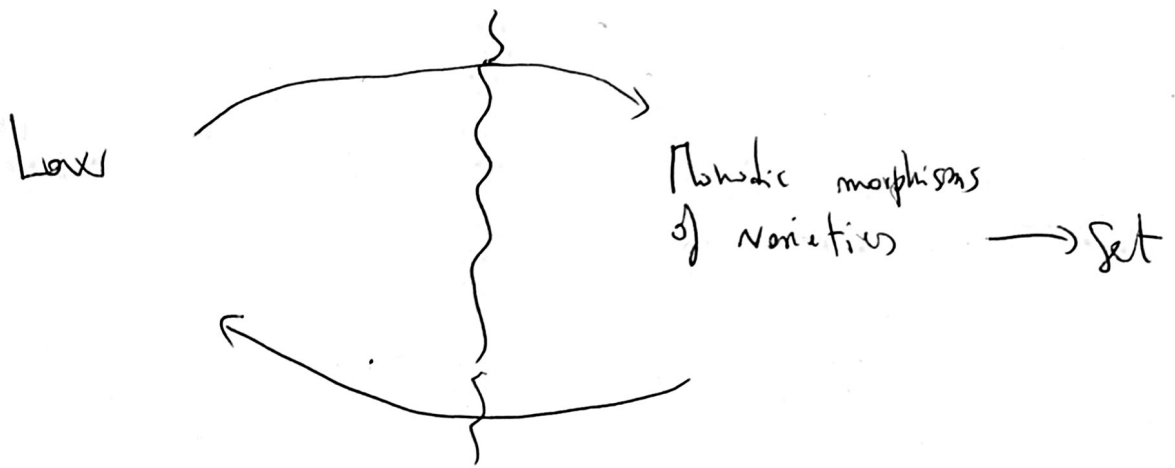
$$\mathcal{W} \xrightarrow{f_*} \text{Set}, \text{ then}$$

~~the~~ the reduced map $\text{Fin} \xrightarrow{f^*} \mathcal{W}_{\text{up}}$ is easily bo.



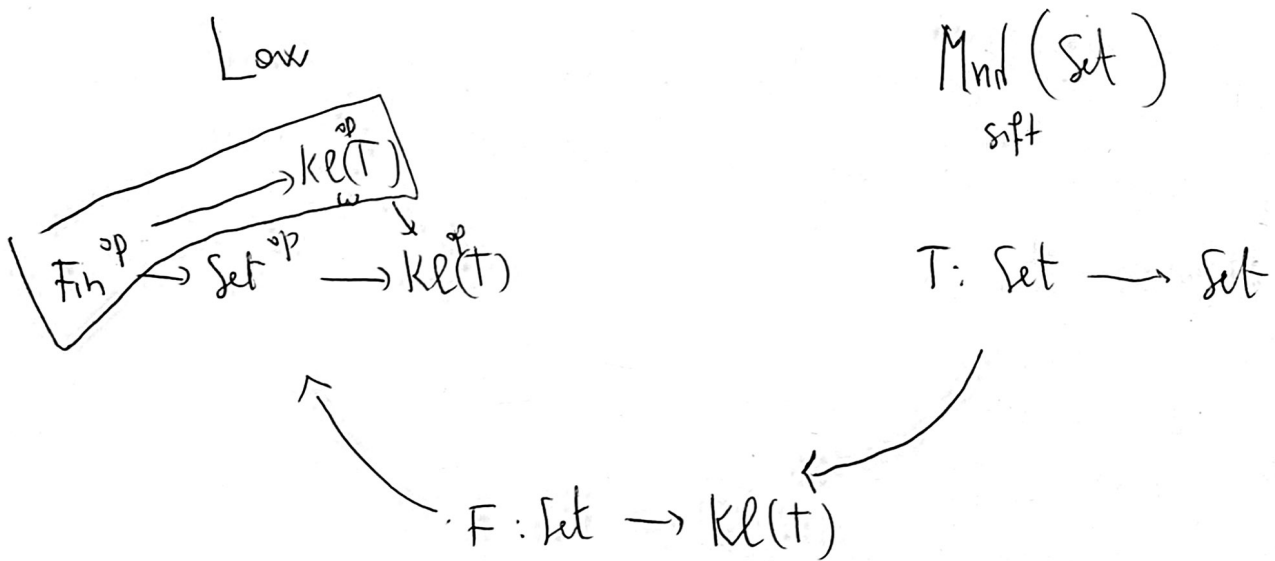
So the construction

$$\text{Fin} \longrightarrow \mathcal{W}_{\text{up}} \text{ is bo.}$$



this is the pic we got.

Of course, a monadic morphism $\mathcal{W} \xrightarrow{f} \text{Set}$ such that f_* preserve sifted colimits is the same as a monad on Set preserving sifted colimits.



$$\text{Fin}^{\text{op}} \longrightarrow \mathcal{L}$$



$$\text{Mnd}(\mathcal{L}) \begin{matrix} \xleftarrow{F^*} \\ \xrightarrow{f_*} \end{matrix} \text{Set}$$

$$\boxed{f_x^* f_x : \text{Set} \longrightarrow \text{Set}}$$



Prop On the category of sets, a filtered presheaf has sifted colimits iff it has direct colimits.

$$\text{Mod}(\text{Set})_{\text{filt}} \simeq \text{Mod}(\text{Set})_{\text{sift}}$$



Morphisms of Monads

So we discussed a correspondence between



But what about the morphisms?!

For Lex theories we have natural transformations.

For monads?

Def A morphism of monads $T \xrightarrow{\varphi} S$ is a natural transformation which is a morphism of monads.

Easy to see that morphism of Alts is
is to morphisms of monads.