# L8 Gabriel-Ulmer duality

This lecture closes the module on sketches, which we end with a syntax-semantics duality for limit-theories. We shall also comment on the reasons why mixed sketches do not have such a ~~sat~~ satisfying duality and its geometric relevance.

In the previous episode we saw a very important bit of G-U duality.

**thm** (Representation) Let $\mathcal{K}$ be a locally $\lambda$-presentable category. Then we have an equivalence

$$\mathcal{K} \simeq \mathrm{Cont}_\lambda(\mathcal{K}_\lambda^{op}, \mathrm{Set}).$$

In particular this theorem tells us that l-p. cats are sketchable via a limit sketch.

We also saw (but we shall recap it now) that every category of models of a limit sketch is locally presentable. Let's recall the proof.

**thm** Let $\mathcal{C}$ be a category w/ $\lambda$-small limits. Then $\mathrm{Cont}_\lambda(\mathcal{C}, \mathrm{Set})$ is locally $\lambda$-presentable.

**Proof** Step 1 $\quad \mathrm{Cont}_\lambda(\mathcal{C}, \mathrm{Set}) \overset{i}{\hookrightarrow} \mathrm{Set}^{\mathcal{C}}$

- $i$ preserves all limits,
- $\lambda$-filtered colimits
- is fully faithful.

Step 2 . $i$ has a left adjoint $L$, constructed by observing that there is a small set of maps $M$ such that

$$\mathrm{Cont}_\lambda(\mathcal{C}, \mathrm{Set}) = M^{\perp}$$

and then using the small object argument that freely adds the witnesses of the orthogonality.

Step 3

$$\mathcal{C}^{op} \xrightarrow{f^{\#}} \mathrm{Set}^{\mathcal{C}} \xrightarrow{L} \mathrm{Cont}_\lambda(\mathcal{C}, \mathrm{Set})$$

the essential image of the composition above form a strong (actually dense) generator made by $\lambda$-presentable objects, indeed

$$\mathrm{Cont}_\lambda(\mathcal{C}, \mathrm{Set})(Lf^{\#}, -) \simeq \mathrm{Set}^{\mathcal{C}}(f^{\#}-, i-)$$

$f^{\#}(c)$ is tiny and $i-$ preserve $\lambda$-filtered colimits.

Cor Locally $\lambda$-pres categories are exactly those of the form $\mathrm{Cont}_\lambda(\mathcal{C}, \mathrm{Set})$.

Now, GU duality completes this theorem, framing it in a more elegant and more informative duality.

<u>Thm</u> (GU) There is a biequivalence of 2-categories

$$\begin{tikzcd} \mathrm{Lex}^{op} \arrow[rr, bend right, "\mathrm{Mod}"'] \arrow[rr, phantom, "\perp"] & & \mathrm{LfP} \arrow[ll, bend right, "\mathrm{th}"'] \end{tikzcd}$$

So, before proving this theorem, let's inspect it parts.

<u>On the left we have Lex</u>

$$\mathrm{Lex}\begin{cases} \text{obj} = \text{categories w finite limits} \\ \text{mor} = \text{functors preserving them.} \end{cases}$$

Of course Lex could be easily replaced with $\lambda$-Lex, and we shall ignore the 2-dimensional part of the statement as it contains no surprise.

<u>On the righ we have Lfp</u>

$$\underline{\mathrm{Lfp}}\begin{cases} \text{obj} = \text{lfp categories} \\ \text{mor} = \text{right adjoints preserving filtered colimits.} \end{cases}$$

Yes, this lecture is very reminiscent of $\boxed{L4}$ because they are variations of the same duality. This time we will focus on a couple of subtleties.

$\boxed{\text{Mod}}$ : $\mathrm{Lex}^{op} \longrightarrow \mathrm{Lfp}$

takes $\mathcal{C}$ and maps it to $\mathrm{Lex}(\mathcal{C}, \mathrm{Set})$
and for a functor $\mathcal{C} \xrightarrow{f} \mathcal{D}$ we map it to

$$\mathrm{Lex}(\mathcal{D}, \mathrm{Set}) \xrightarrow{f^*} \mathrm{Lex}(\mathcal{C}, \mathrm{Set})$$

which is continuous, preserve directed colimits and has a left adjoint by SOA or AFT.

$$\begin{tikzcd}
\mathrm{Lex}(\mathcal{D}, \mathrm{Set}) \arrow[r, "f^*"] \arrow[d, hook] & \mathrm{Lex}(\mathcal{C}, \mathrm{Set}) \arrow[d, hook] \\
\mathrm{Set}^{\mathcal{D}} \arrow[r, "f^*"] & \mathrm{Set}^{\mathcal{C}}
\end{tikzcd}$$

filt colts

all colimits

creates filt colts.

$\boxed{\text{Th}}$ : $\mathrm{Lfp} \longrightarrow \mathrm{Lex}^{op}$

As usual this has two presentations, either we choose to send $\mathcal{K}$ to $\mathcal{K}^{op}_{\omega}$ or to

$\mathrm{Lfp}(\mathcal{K}, \mathrm{Set})$, indeed

$$\mathcal{K}_\omega^{op} \simeq \mathrm{Lfp}(\mathcal{K}, \mathrm{Set})$$

(the second choice make the functoriality more transparent).

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Unit and Counit of the adjunction

($\eta$)

$$\mathcal{K} \xrightarrow{\eta} \mathrm{Mod\,th}(\mathcal{K})$$

$\eta$ is nothing but the "representation theorem"

$$\mathcal{K} \simeq \mathrm{Lex}(\mathcal{K}_\omega^{op}, \mathrm{Set})$$

($\varepsilon$)

$$\begin{tikzcd}
 & \mathcal{C}^{op} \arrow[dl] \arrow[dr, dashed, "\varepsilon"] & \\
\mathrm{Lex}(\mathcal{C}, \mathrm{Set}) & & \mathrm{Lex}(\mathcal{C}, \mathrm{Set})_\omega \arrow[ll, hook']
\end{tikzcd}$$

$\varepsilon$ is "the fact that the Yoneda embedding lands in finitely presentable objects.

Rem Differently from the duality $\mathrm{Prod} \rightleftarrows \mathrm{Var}$ in this case $\varepsilon$ is an equivalence because $\mathcal{C}$ is already Cauchy complete!

## The general paradigm of syntactic categories.

So, in the first model we met universal algebra, or "product theories". Then, at a very high level of expressivity we met sketches.

$$\text{Prod} \qquad \text{Lex} \qquad \underbrace{\qquad \cdots \qquad} \qquad \text{Sketches}$$

universal algebra

Lex can be understood via "partial universal algebra", in the sense that operations are not globally defined (Pos, Cat). But in the gap between Lex and sketches there is a plethora of fragments of logic that we shall not study in detail.

$$\text{Prod} \quad \text{Lex} \quad \underbrace{\text{Disj} \quad \text{Reg} \quad \text{Ex} \quad \text{Coh} \quad \text{Geom}} \quad \text{Sketches}$$

Disjunctive Regular exact coherent geometric

These fragment of (first order) logic correspond (categorically) to special families of mixed sketches

Reg for example is made of "finite limit" sketches, plus the prescription of some epimorphic maps.

$Set_{>0}$ is an example of a category in regular logic

The question of syntax-semantic duality can be raised for these fragments of logic too. The serious issue is that we lose our "cute models". Let's see what I mean by that.

For $\mathcal{C}$ a category with finite limits, we have a functor

$$\mathcal{C}^{op} \xrightarrow{\text{よ}^{\#}} \mathrm{Lex}(\mathcal{C}, \mathrm{Set})$$

that not only gives us for free some models of the theory, but gives us a <u>dense</u> family of models, that is of course enough to recover the whole theory.

In the case of regular logic (for example) we already saw that this is not often the case.

$$\mathbb{T} = \left(\cdot, \emptyset, \text{"force } \cdot \text{ to be initial"}\right)$$

$$\begin{tikzcd} & \bullet \arrow[dl, dashed, "\times"'] \arrow[d, "\text{よ}^{\#}"] \\ \mathrm{Mod}(\mathbb{T}) \arrow[r, hook] & \mathrm{Set} \end{tikzcd}$$

clearly $\text{よ}^{\#}(\cdot)$ is not a model of the theory.

thus, for more complex dualities the category of models

$$(\mathrm{Mod}(\mathbb{T}), \overset{?}{\ldots})$$

must be ~~e~~ decorated with more information which will allow to recover the theory.

In this course we will not focus in detail on each of these fragments, but we will treat geometric logic, which "swallows" regular, coherent and others and see how to partially fix this issue for coherent logic.