L9 Hyperdoctrines

In the previous lectures we have used <u>sketches</u> to specify theories. Their "functorial semantic" offers then the theory of their models.
In this lecture we present a different approach to specify a theory, the general notion of <u>doctrine</u>. So, doctrines are alternative ways to present (first order) theories.

The theory of doctrine "starts" in 1847, with the introduction of <u>Boolean algebras</u>.

(1) <u>Boole</u>

~~The~~ The entrypoint for modern mathematical logic was the introduction of Boolean algebras, or more generally posets with enough structure to simulate linguistic operators in the ~~the~~ natural discourse

$$\underbrace{\varphi}_{\text{proposition}} \qquad \overset{\text{top}}{\underbrace{\top}_{\text{truth values}}}\ \overset{\text{bottom}}{\bot} \qquad \underbrace{\neg \quad \vee \quad \wedge}_{\text{connectives}}$$

For a while, this program was known as "algebraization of logic", and indeed its purpose was clear.
Propositions are just symbols, what we truly care about is the way we algebraically manipulate them

$$\boxed{\varphi \wedge \psi}\,.$$

Now, Boole, and later Stone, were very much aware that the Boolean algebra that makes statement about a set is a "very specific one". After Zermelo we understand that as the powerset of X

$$(\mathcal{P}(X), \vee, \wedge, \neg, \top, \bot).$$

The sense in which a formula $\boxed{\varphi}$ identifies a property over X is clarified by the Grothendieck construction.

$\boxed{\text{Yet}}$, when we just write φ we may lose track of the fact that it predicates about X.

$X \vdash \varphi$ $\qquad\qquad$ $x_1 \ldots\ldots x_n \vdash \varphi$

This is the notion of "context", and proposition are given ($\vdash$) in a certain context (the variables appearing in the formula).

Back to theories

Now, we intuitively know that to specify a theory should be about presenting some set of axioms that we shall obey. So some formulas should be there, ready to be "evaluated" true, false, or something.

And indeed a boolean algebra B does that job. But in "interesting" mathematics, we are not content with formulas without variables. So we need a notion of Boolean algebra that is sensitive to the change of context. This is the notion of doctrine.

Def A (primitive) doctrine is a functor

$$\mathcal{P} : \mathrm{Fin}^{op} \longrightarrow \mathrm{Lat}$$

Lat: Posets with finite meets

Mathematics	Logical intuition
$\varphi \in P(X)$	$\dfrac{X \vdash \varphi \quad \text{Proposition}}{x_1 \dots x_n \vdash \varphi(x_1 \dots x_n)}$
$f : X \longrightarrow Y$ $f^* : P(Y) \longrightarrow P(X)$	$\dfrac{y_1 \dots y_n \vdash \varphi(y_1 \dots y_n)}{x_1 \dots x_m \vdash \varphi(fx_1, \dots, fx_m)}$. ⚠ substitution of variables.

two examples

Ex1 We start with a "toy example" that will set the stage for further development. On the category of sets, we can put the "expected" doctrine structure

$$\begin{aligned} \mathcal{P} : \mathrm{Set}^{op} &\longrightarrow \mathrm{Bool} \\ X &\longmapsto \mathcal{P}(X). \end{aligned}$$

Many things should be said.

(1) Set^{op} is not Fin^{op}. The def of doctrine I gave is very limitary. But if we are more flexible with the domain, we can retain the same intuition.

(2) $\mathcal{P}(X)$ is <u>much more</u> than a lattice, and in fact it has the structure of a boolean algebra. So, for more structured/rich theories, we may want to <u>enhance</u> that to something more expressive like Heyting.

(3) Consider a function $f : X \longrightarrow Y$. Then, we have

$$\begin{tikzcd} \mathcal{P}(Y) \arrow[rr, "f^{-1}"] & & \mathcal{P}(X) \arrow[ll, bend right, "f_!"'] \arrow[ll, bend left, "f_*"] \end{tikzcd}$$

[this is more evident when $Y = 1$.]

$$f_! \varphi \geq \psi \iff \varphi \geq f^{-1}\psi$$

$$\frac{X \vdash \varphi}{Y \vdash f_! \varphi}$$

$$\boxed{\exists_f \varphi} > \psi \iff \varphi > \psi(f-)$$

this phenomenon was called "adjointness in foundation" and it amounts to the fact that

$$\boxed{\forall_x \varphi} \quad \text{and} \quad \boxed{\exists_x \varphi}$$

provide left and right adjoint to f^*.

<u>Ex2</u> (Only for the logicians in the room).

Let $\mathbb{T}$ be a first order theory in a first order language $\mathcal{L}$.

~~[illegible]~~

~~[illegible]~~

then we have the Lindenbaum-Tarski algebra of well formed formulas

$$\mathcal{P}_{\mathbb{T}}(X) = \left\{ [\varphi(x_1 \ldots x_n)] \;\; \text{formulas in } \begin{matrix} x \\ \text{variables} \end{matrix} \right\}.$$

Notice that this has a lot of properties!

<u>The notion of model</u> / <u>functorial semantics</u>

To understand the notion of model of a doctrine, we may go back to Boole

A model of a boolean algebra is an assignment of truth values for its propositions

$$B \xrightarrow{[\;]} \{0 < 1\}$$

which of course respects the algebraic structure.

Of course, the notion of model for ~~a~~ doctrine should be the same, but keeping into account that we must compose contexts correctly

$$\mathcal{P}(\underline{X}) \rightsquigarrow \mathcal{Q}(\underline{X})$$

<u>Def</u> Let $(\mathcal{C}, \mathcal{P})$, $(\mathcal{D}, \mathcal{Q})$ be doctrines. A morphism of doctrines is a lex functor (f) and a natural transformation (λ)

$$\begin{tikzcd} \mathcal{C}^{op} \arrow[rr, "f^{op}"] \arrow[dr, "\mathcal{P}"'] & \arrow[d, phantom, "\xRightarrow{\lambda}"] & \mathcal{D}^{op} \arrow[dl, "\mathcal{Q}"] \\ & \mathrm{Lat} & \end{tikzcd}$$

this amounts to

$$\boxed{\mathcal{P}(c) \xrightarrow{\lambda} \mathcal{Q}(f(c))}$$

A model of $(\mathcal{C}, \mathcal{P})$ is a morphism into $\underline{(\mathrm{Set}, \mathcal{P})}$.

Caveats

- Wait a second. $\mathcal{C}, \mathcal{D}$ are not fin! how come? As a said there is no reason to stay with fin. $\mathcal{C}$ is just a category with finite limits.
- the choice of $(\mathrm{Set}, \mathcal{P})$ is both supported by experience, but also clear by the usual functional semantics