

L9 Hyperdoctrines

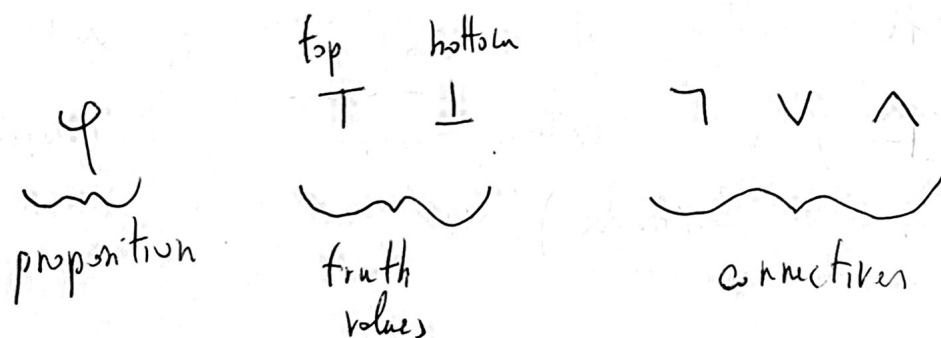
In the previous lectures we have used sketches to specify theories. Their "functional semantic" offers them the theory of their models.

In this lecture we present a different approach to specify a theory, the general notion of doctrine. So, doctrines are alternative ways to present (first order) theories.

The theory of doctrine "starts" in 1847, with the introduction of Boolean algebras.

(1) Boole

~~The~~ The entrypoint for modern mathematical logic was the introduction of Boolean algebras, or more generally posets with enough structure to simulate linguistic operators in the natural discourse.



For a while, this program was known as "algebraization of logic", and indeed its purpose was clear.

Propositions are just symbols, what we truly care about is the way we algebraically manipulate them

$$\boxed{\varphi \wedge \psi}.$$

Now, Boole, and later Stone, were very much aware that the Boolean algebra that makes statement about a set is a "very specific one". After some time we understand that as the power set of X

$$(P(X), \cup, \cap, \tau, \perp, \perp)$$

The sense in which a formula φ identifies a property over X is identified by the Gödel construction.

Yet, when we just write φ we may lose track of the fact that it predicates about X .

$$X \vdash \varphi \qquad x_1, \dots, x_n \vdash \varphi$$

This is the notion of "context" and proposition are given (\vdash) in a certain context (the variables appearing in the formula).

Back to theories

Now, we intuitively know that to specify a theory should be about presenting some set of axioms that we shall obey. So some formulas should be there, ready to be "evaluated" true, false, or something.

And indeed a boolean algebra B does that job.
 But in "interesting" mathematics, we are not content
 with formulas without variables. So we need a notion
 of Boolean algebra that is sensitive to the change
of context. This is the notion of doctrine.

Def

A (primitive) doctrine is a Functor

$$\mathcal{P} : \text{Fin}^{\text{op}} \rightarrow (\text{Lat})$$

Posets with
finite meets

Mathematics

Logical intuition

$$\varphi \in \mathcal{P}(X)$$

$$X \vdash \varphi \text{ Proposition}$$

$$x_1, \dots, x_n \vdash \varphi(x_1, \dots, x_n)$$

$$f : X \rightarrow Y$$

$$y_1, \dots, y_n \vdash \varphi(y_1, \dots, y_n)$$

$$f^* : \mathcal{P}(Y) \rightarrow \mathcal{P}(X)$$

$$x_1, \dots, x_m \vdash \varphi(fx_1, \dots, fx_m)$$

substitution
of variables.

Two examples

Ex 1 We start with a "bad example" that will set the stage for further development. On the category of sets, we can put the "expected" lattice structure

$$\mathcal{P} : \text{Set}^{\text{op}} \longrightarrow \text{Bool}$$

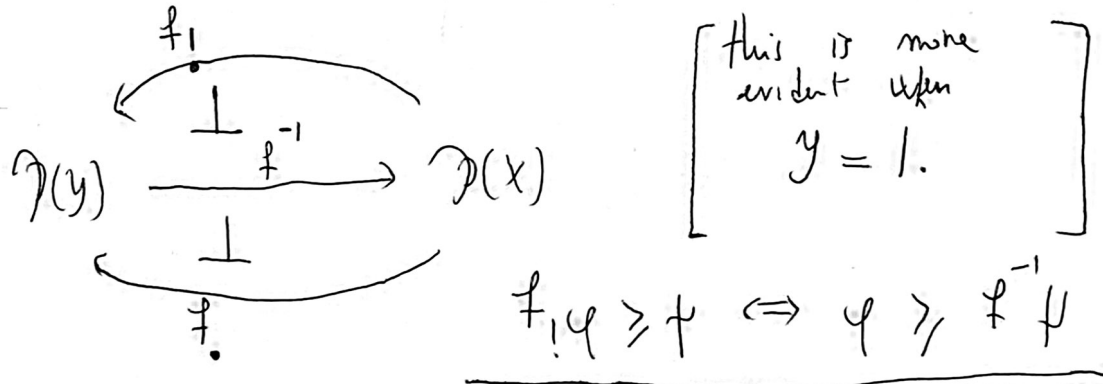
$$X \longmapsto \mathcal{P}(X)$$

Many things should be said:

(1) Set^{op} is not Fin^{op} . The def of lattice I gave is very finitary. But if we are more flexible with the domain, we can recover the same intuition.

(2) $\mathcal{P}(X)$ is much more than a lattice, and in fact it has the structure of a boolean algebra. So, for more structured/rich theories, we may want to enhance lat to something more expressive like Heyting.

(3) Consider a function $f: X \rightarrow Y$. Then, we have



$$f_! \varphi \geq \psi \iff \varphi \geq f^{-1} \psi$$

$$\frac{X \vdash \varphi}{Y \vdash f_! \varphi}$$

$$\boxed{\exists f \varphi} \vdash \psi \iff \varphi \vdash f^{-1} \psi$$

this phenomenon was called "adjointness in foundation" and it amounts to the fact that

$$\boxed{\forall x \varphi} \quad \text{and} \quad \boxed{\exists x \varphi}$$

provide left and right adjoint to f^* .

Ex 2 (Only for the logicians in the room).

Let Π be a first order theory in a first order language \mathcal{L} .

~~we have a category~~

~~of models~~

then we have the Lindenbaum-Tarski algebra of well formed formulas

$$\mathcal{P}_+(X) = \{ [\varphi(x_1, \dots, x_n)] \text{ formulas in } x \text{ variables} \}.$$

Notice that this has a lot of properties!

the notion of model / functional semantics

To understand the notion of model of a doctrine, we may go back to Boolean

A model of a boolean algebra is an assignment of truth values for its propositions

$$B \xrightarrow{[\]} \{0 < 1\}$$

which of course respects the algebraic structure.

Of course, the notion of model for a doctrine should be the same, but keeping into account that we must compare contexts correctly

$$\mathcal{P}(x) \rightsquigarrow \mathcal{Q}(x)$$

Def Let $(\mathcal{C}, \mathcal{P})$, $(\mathcal{D}, \mathcal{Q})$ be doctrines. A morphism of doctrines is a lex functor (f) and a natural transformation (η)

$$\begin{array}{ccc} \mathcal{C}^{\text{op}} & \xrightarrow{f} & \mathcal{D}^{\text{op}} \\ \mathcal{P} \searrow & \eta \rightrightarrows & \searrow \mathcal{Q} \\ & \text{Set} & \end{array}$$

this amounts to

$$\boxed{\mathcal{P}(c) \xrightarrow{\eta} \mathcal{Q}(f(c))}$$

A model of $(\mathcal{C}, \mathcal{P})$ is a morphism into (Set, \mathcal{P})

Comments

- Went a round. \mathcal{C}, \mathcal{D} are not fin^{op} ! how come?
As a Sayed fin is no need to stay with fin .
 \mathcal{C} is just a category with finite limits.
- The choice of $(\text{Set}, \mathcal{P})$ is both supported by existence, but also clear by the usual functorial semantics